ON CERTAIN ANALYTIC UNIVALENT FUNCTIONS

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ABSTRACT. We consider the class of analytic functions $B(\alpha)$ to investigate some properties for this class. The angular estimates of functions in the class $B(\alpha)$ are obtained. Finally, we derive some interesting conditions for the class of strongly starlike and strongly convex of order $\alpha$ in the open unit disk.

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1. Introduction. Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function $f(z)$ belonging to $A$ is said to be starlike of order $\alpha$ if it satisfies

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $S_\alpha^*$ the subclass of $A$ consisting of functions which are starlike of order $\alpha$ in $U$. Also, a function $f(z)$ belonging to $A$ is said to be convex of order $\alpha$ if it satisfies

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $C_\alpha$ the subclass of $A$ consisting of functions which are convex of order $\alpha$ in $U$.

If $f(z) \in A$ satisfies

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$, then $f(z)$ said to be strongly starlike of order $\alpha$ in $U$, and this class denoted by $S_\alpha^{**}$. If $f(z) \in A$ satisfies

$$\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$, then we say that $f(z)$ is strongly convex of order $\alpha$ in $U$, and we denote by $C_\alpha^*$ the class of all such functions.
The object of the present paper is to investigate various properties of the following class of analytic functions defined as follows.

**Definition 1.1.** A function \( f(z) \in A \) is said to be a member of the class \( B(\alpha) \) if and only if

\[
\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha
\]

(1.6)

for some \( 0 \leq \alpha < 1 \) and for all \( z \in U \).

Note that condition (1.6) implies

\[
\text{Re} \left( \frac{z^2 f'(z)}{f^2(z)} \right) > \alpha.
\]

(1.7)

2. **Main results.** In order to derive our main results, we have to recall here the following lemmas.

**Lemma 2.1** (see [2]). Let \( f(z) \in A \) satisfy the condition

\[
\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 \quad (z \in U),
\]

(2.1)

then \( f \) is univalent in \( U \).

**Lemma 2.2** (see [1]). Let \( w(z) \) be analytic in \( U \) and such that \( w(0) = 0 \). Then if \( |w(z)| \) attains its maximum value on circle \( |z| = r < 1 \) at a point \( z_0 \in U \), we have

\[
z_0 w'(z_0) = kw(z_0),
\]

(2.2)

where \( k \geq 1 \) is a real number.

**Lemma 2.3** (see [3]). Let a function \( p(z) \) be analytic in \( U \), \( p(0) = 1 \), and \( p(z) \neq 0 \) \( (z \in U) \). If there exists a point \( z_0 \in U \) such that

\[
|\arg(p(z))| < \frac{\pi}{2} \alpha, \quad \text{for } |z| < |z_0|, \quad |\arg(p(z_0))| = \frac{\pi}{2} \alpha,
\]

(2.3)

with \( 0 < \alpha \leq 1 \), then we have

\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik \alpha,
\]

(2.4)

where

\[
k \geq \frac{1}{2} \left( \frac{1}{a} + 1 \right) \geq 1 \quad \text{when } \arg(p(z_0)) = \frac{\pi}{2} \alpha,
\]

\[
k \leq -\frac{1}{2} \left( \frac{1}{a} + 1 \right) \leq -1 \quad \text{when } \arg(p(z_0)) = -\frac{\pi}{2} \alpha,
\]

(2.5)

\[
p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0).
\]

We begin with the statement and the proof of the following result.
**Theorem 2.4.** If \( f(z) \in A \) satisfies
\[
\left| \frac{(zf(z))'' - 2zf'(z)}{f'(z)} \right| < \frac{1 - \alpha}{2 - \alpha} \quad (z \in U),
\]
for some \( \alpha \) (\( 0 \leq \alpha < 1 \)), then \( f(z) \in B(\alpha) \).

**Proof.** We define the function \( w(z) \) by
\[
\frac{z^2 f'(z)}{f^2(z)} = 1 + (1 - \alpha) w(z).
\]
Then \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). By the logarithmic differentiations, we get from (2.7) that
\[
\frac{(zf(z))'' - 2zf'(z)}{f'(z)} \frac{1}{f(z)} = \frac{(1 - \alpha) z w'(z)}{1 + (1 - \alpha) w(z)}.
\]

Suppose there exists \( z_0 \in U \) such that
\[
\max_{|z| = |z_0|} |w(z)| = |w(z_0)| = 1,
\]
then from Lemma 2.2, we have (2.2).

Letting \( w(z_0) = e^{i\theta} \), from (2.8), we have
\[
\frac{(z_0 f(z_0))'' - 2z_0 f'(z_0)}{f'(z_0)} = \frac{(1 - \alpha) ke^{i\theta}}{1 + (1 - \alpha)e^{i\theta}} \approx \frac{1 - \alpha}{2 - \alpha},
\]
which contradicts our assumption (2.6). Therefore \( |w(z)| < 1 \) holds for all \( z \in U \). We finally have
\[
\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| = (1 - \alpha) |w(z)| < 1 - \alpha \quad (z \in U),
\]
that is, \( f(z) \in B(\alpha) \).

Taking \( \alpha = 0 \) in Theorem 2.4 and using Lemma 2.1 we have the following corollary.

**Corollary 2.5.** If \( f(z) \in A \) satisfies
\[
\left| \frac{(zf(z))'' - 2zf'(z)}{f'(z)} \right| < \frac{1}{2} \quad (z \in U),
\]
then \( f \) is univalent in \( U \).

Next, we prove the following theorem.

**Theorem 2.6.** Let \( f(z) \in A \). If \( f(z) \in B(\alpha) \), then
\[
\left| \arg \left( \frac{f(z)}{z} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U),
\]
for some \( \alpha \) (\( 0 < \alpha < 1 \)) and \((2/\pi) \tan^{-1} \alpha - \alpha = 1\).
Proof. We define the function \( p(z) \) by
\[
\frac{f(z)}{z} = p(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1}.
\] (2.14)

Then we see that \( p(z) \) is analytic in \( U \), \( p(0) = 1 \), and \( p(z) \neq 0 \) \( (z \in U) \). It follows from (2.14) that
\[
\frac{z^2 f'(z)}{f^2(z)} = \frac{1}{p(z)} \left( 1 + z p'(z) \right).
\] (2.15)

Suppose there exists a point \( z_0 \in U \) such that
\[
\left| \arg(p(z)) \right| < \frac{\pi}{2} \alpha, \text{ for } |z| < |z_0|, \quad |\arg(p(z_0))| = \frac{\pi}{2} \alpha.
\] (2.16)

Then, applying Lemma 2.3, we can write that
\[
\frac{z_0 p'(z_0)}{p(z_0)} = i k \alpha,
\] (2.17)

where
\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \geq 1 \text{ when } \arg(p(z_0)) = \frac{\pi}{2} \alpha,
\]
\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \leq -1 \text{ when } \arg(p(z_0)) = -\frac{\pi}{2} \alpha,
\] (2.18)

\[p(z_0)^{1/\alpha} = \pm ai, \quad (a > 0).\]

Therefore, if \( \arg(p(z_0)) = \pi \alpha/2 \), then
\[
\frac{z_0^2 f'(z_0)}{f^2(z_0)} = \frac{1}{p(z_0)} \left( 1 + z_0 p'(z_0) \right) = a^{-\alpha} e^{-i \pi \alpha/2} (1 + ik \alpha).
\] (2.19)

This implies that
\[
\arg\left( \frac{z_0^2 f'(z_0)}{f^2(z_0)} \right) = \arg\left( \frac{1}{p(z_0)} \left( 1 + z_0 p'(z_0) \right) \right)
\]
\[
= -\frac{\pi}{2} \alpha + \arg(1 + i \alpha k) \geq -\frac{\pi}{2} \alpha + \tan^{-1} \alpha \quad (2.20)
\]
\[
= \frac{\pi}{2} \left( \frac{2}{\pi} \tan^{-1} \alpha - \alpha \right) = \frac{\pi}{2}
\]

if
\[
\frac{2}{\pi} \tan^{-1} \alpha - \alpha = 1.
\] (2.21)

Also, if \( \arg(p(z_0)) = -\pi \alpha/2 \), we have
\[
\arg\left( \frac{z_0^2 f'(z_0)}{f^2(z_0)} \right) \leq -\frac{\pi}{2}
\] (2.22)

if
\[
\frac{2}{\pi} \tan^{-1} \alpha - \alpha = 1.
\] (2.23)

These contradict the assumption of the theorem.
Thus, the function \( p(z) \) has to satisfy
\[
| \arg(p(z)) | < \frac{\pi}{2} \alpha \quad (z \in U)
\]  
(2.24)

or
\[
| \arg\left( \frac{f(z)}{z} \right) | < \frac{\pi}{2} \alpha \quad (z \in U).
\]  
(2.25)

This completes the proof. \( \Box \)

Now, we prove the following theorem.

**Theorem 2.7.** Let \( p(z) \) be analytic in \( U \), \( p(z) \neq 0 \) in \( U \) and suppose that
\[
\left| \arg\left( p(z) + \frac{z^3 f'(z)}{f^2(z)} p'(z) \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U),
\]  
(2.26)

where \( 0 < \alpha < 1 \) and \( f(z) \in B(\alpha) \), then we have
\[
| \arg(p(z)) | < \frac{\pi}{2} \alpha \quad (z \in U).
\]  
(2.27)

**Proof.** Suppose there exists a point \( z_0 \in U \) such that
\[
| \arg(p(z_0)) | < \frac{\pi}{2} \alpha, \quad \text{for } |z| < |z_0|, \quad | \arg(p(z_0)) | = \frac{\pi}{2} \alpha.
\]  
(2.28)

Then, applying Lemma 2.3, we can write that
\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik \alpha,
\]  
(2.29)

where
\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg(p(z_0)) = \frac{\pi}{2} \alpha,
\]
\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg(p(z_0)) = -\frac{\pi}{2} \alpha,
\]  
(2.30)

\[
p(z_0)^{1/\alpha} = \pm a i, \quad (a > 0).
\]

Then it follows that
\[
\arg\left( p(z_0) + \frac{z_0^3 f'(z_0)}{f^2(z_0)} p'(z_0) \right) = \arg\left( p(z_0) \left( 1 + \frac{z_0^2 f'(z_0) z p'(z_0)}{f^2(z_0) p(z_0)} \right) \right)
\]
\[
= \arg\left( p(z_0) \left( 1 + i \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k \right) \right).
\]  
(2.31)

When \( \arg(p(z_0)) = \pi \alpha/2 \), we have
\[
\arg\left( p(z_0) + \frac{z_0^3 f'(z_0)}{f^2(z_0)} p'(z_0) \right) = \arg(p(z_0)) + \arg\left( 1 + i \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k \right) > \frac{\pi}{2} \alpha,
\]  
(2.32)

because
\[
\text{Re} \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k > 0 \text{ and therefore } \arg\left( 1 + i \frac{z_0^2 f'(z_0)}{f^2(z_0)} \alpha k \right) > 0.
\]  
(2.33)
Similarly, if \( \arg(p(z_0)) = -\pi \alpha / 2 \), then we obtain that

\[
\arg \left( p(z_0) + \frac{z_0^3 f'(z_0)}{f''(z_0)} p'(z_0) \right) = \arg(p(z_0)) + \arg \left( 1 + i \frac{z_0^2 f'(z_0)}{f''(z_0)} \alpha k \right) < -\frac{\pi}{2} \alpha, \quad (2.34)
\]

because

\[
\Re \frac{z_0^2 f'(z_0)}{f''(z_0)} \alpha k < 0 \quad \text{and therefore} \quad \arg \left( 1 + i \frac{z_0^2 f'(z_0)}{f''(z_0)} \alpha k \right) < 0. \quad (2.35)
\]

Thus we see that (2.32) and (2.34) contradict our condition (2.26). Consequently, we conclude that

\[
\left| \arg(p(z)) \right| < \frac{\pi}{2} \alpha \quad (z \in U). \quad (2.36)
\]

Taking \( p(z) = z f'(z) / f(z) \) in Theorem 2.7, we have the following corollary.

**Corollary 2.8.** If \( f(z) \in A \) satisfying

\[
\left| \arg \left( \frac{z f'(z)}{f(z)} + \frac{z^3 f'(z)}{f^4(z)} \left( (z f'(z))' - z (f'(z))^2 \right) \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U), \quad (2.37)
\]

where \( 0 < \alpha < 1 \) and \( f(z) \in B(\alpha) \), then \( f(z) \in \bar{S}_\alpha^* \).

Taking \( p(z) = 1 + z f''(z) / f'(z) \) in Theorem 2.7, we have the following corollary.

**Corollary 2.9.** If \( f(z) \in A \) satisfying

\[
\left| \arg \left( \frac{(z f'(z))^3}{f'(z)} + \frac{z^3}{f^3(z)} \left( (z f''(z))' - z (f''(z))^2 \right) \right) \right| < \frac{\pi}{2} \alpha, \quad (2.38)
\]

where \( 0 < \alpha < 1 \) and \( f(z) \in B(\alpha) \), then \( f(z) \in \bar{C}_\alpha \).

**References**


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