ON SOME CLASSES OF BCH-ALGEBRAS

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ABSTRACT. The concept of a BCH-algebra is a generalization of the concept of a BCI-algebra. It is shown that weakly commutative BCH-algebras are weakly commutative BCI-algebras. Moreover, the concepts of weakly positive implicative and weakly implicative BCH-algebras are defined and it is shown that every weakly implicative BCH-algebra is a weakly positive implicative BCH-algebra. The weakly positive implicative BCH-algebras are characterized with the help of their self maps. Two open problems are posed.

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1. Introduction. In 1966, Imai and Iséki introduced two classes of abstract algebras, BCK-algebras and BCI-algebras [6, 7]. BCI-algebras are a generalization of BCK-algebras. These algebras have been extensively studied since their introduction. In 1983, Hu and Li [4, 5] introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK- and BCI-algebras. They have studied a few properties of these algebras. Certain other properties have been studied by Chaudhry [2] and Dudek and Thomys [3]. It has been shown [3, 4, 5] that there are no proper associative and medial BCH-algebras, that is, associative and medial BCH-algebras are associative and medial BCI-algebras, respectively.

The purpose of this paper is to investigate the existence of certain classes of proper BCH-algebras and study their properties. It is shown that proper weakly commutative BCH-algebras do not exist. However, proper weakly positive implicative and proper weakly implicative BCH-algebras exist and every weakly implicative BCH-algebra is a weakly positive implicative BCH-algebra but not conversely. Weakly positive implicative BCH-algebras have been characterized in terms of their self maps. The results proved in this paper are general in the sense that corresponding results for BCK-algebras and BCI-algebras become special cases.

2. Preliminaries. In this section, we describe certain definitions, known results, and examples that will be used in the sequel.

DEFINITION 2.1 (see [9]). A BCI-algebra is an algebra \((X, *, 0)\) of type \((2, 0)\) satisfying the following conditions:

1. \((x * y) * (x * z) \leq z * y',\)
2. \(x * (x * y') \leq y',\)
3. \(x \leq x,\)
4. \(x \leq y \text{ and } y \leq x \Rightarrow x = y,\)
5. \(x \leq 0 \text{ implies } x = 0, \text{ where } x \leq y \text{ is defined by } x * y = 0.\)
If (5) is replaced by $0 \leq x$, then the algebra is called a BCK-algebra. It is known that every BCK-algebra is a BCI-algebra but not conversely. Further, in a BCI-algebra the identity $(x \ast y) \ast z = (x \ast z) \ast y$ holds [9].

**Definition 2.2** (see [4]). A BCH-algebra is an algebra $(X, \ast, 0)$ of type $(2, 0)$ satisfying the following conditions:

1. $x \leq x$,
2. $x \leq y, y \leq x$ imply $x = y$,
3. $(x \ast y) \ast z = (x \ast z) \ast y$, where $x \leq y$ if and only if $x \ast y = 0$.

In any BCH-algebra, the following hold:

1. $x \ast (x \ast y) \leq y$ [4],
2. $x \ast 0 = 0$ implies $x = 0$ [4],
3. $0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y)$ [3],
4. $x \ast 0 = x$ [3],
5. $(x \ast y) \ast x = 0 \ast y$ [4],
6. $x \leq y$ implies $0 \ast x = 0 \ast y$ [2].

It is known that every BCI-algebra is a BCH-algebra but the following example shows that the converse is not true.

**Example 2.3** (see [4]). Let $X = \{0, 1, 2, 3\}$ in which $\ast$ is defined by:

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Then $(X, \ast, 0)$ is a BCH-algebra but it is not a BCI-algebra because

$$(2 \ast 3) \ast (2 \ast 1) = 2 \ast 0 = 2 \neq 1 \ast 3 = 3. \quad (2.1)$$

**Example 2.4** (see [2]). Let $X = \{0, 1, 2, 3, 4\}$ in which $\ast$ is defined by:

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Routine calculations give that $(X, \ast, 0)$ is a BCH-algebra but it is not a BCI-algebra because

$$(1 \ast 3) \ast (1 \ast 2) = 1 \ast 0 = 1 \neq 2 \ast 3 = 0. \quad (2.2)$$

In the sequel a BCH-algebra will be simply denoted by $X$.

**Definition 2.5** (see [5]). A BCH-algebra $X$ is called proper if it is not a BCI-algebra.
We note that BCH-algebras of Examples 2.3 and 2.4 are proper BCH-algebras.

**Definition 2.6** (see [4]). A BCH/BCI-algebra $X$ is called associative if $(x * y) * z = x * (y * z)$.

**Definition 2.7** (see [3]). A BCH/BCI-algebra $X$ is called medial if $(x * y) * (z * \mu) = (x * z) * (y * \mu)$.

In the sequel, we shall need the following result.

(11) A BCH-algebra $X$ is proper if and only if it does not satisfy (1) (see [4]).

3. Classification of BCH-algebras. It is known that associative BCH-algebras are associative BCI-algebras and medial BCH-algebras are medial BCI-algebras [3, 4]. Thus a natural question arises whether there exist some interesting classes of proper BCH-algebras or not? We show that there exist proper weakly positive implicative BCH-algebras as well as weakly implicative BCH-algebras. Moreover, the class of weakly implicative BCH-algebras is a proper subclass of the class of weakly positive implicative BCH-algebras. However, weakly commutative BCH-algebras are weakly commutative BCI-algebras.

**Definition 3.1** (see [8]). A BCK-algebra $X$ is called positive implicative if $(x * y) * z = (x * z) * (y * z)$. It is called implicative if $x * (y * x) = x$. It is commutative if $x * (x * y) = y * (y * x)$.

It is well known that positive implicative BCI-algebras, implicative BCI-algebras and commutative BCI-algebras are positive implicative BCK-algebras, implicative BCK-algebras and commutative BCK-algebras, respectively [9].

In [1], Chaudhry defined three classes of proper BCI-algebras, namely, weakly positive implicative BCI-algebras, weakly implicative BCI-algebras, and weakly commutative BCI-algebras. He also investigated a few properties of these algebras. We recall these definitions and the following result.

**Definition 3.2** (see [1]). A BCI-algebra $X$ is called weakly positive implicative if

(12) $(x * y) * z = ((x * z) * z) * (y * z)$.

It is called weakly implicative if

(13) $(x * (y * x)) * (0 * (y * x)) = x$.

It is called weakly commutative if

(14) $(x * (x * y)) * (0 * (x * y)) = y * (y * x)$.

**Theorem 3.3** (see [1]). A BCI-algebra $X$ is weakly positive implicative if and only if

(15) $x * y = ((x * y') * y') * (0 * y)$.

We note that Theorem 3.3 tells us that (12) and (15) are equivalent in a BCI-algebra. However, they are not equivalent in a BCH-algebra. We consider the BCH-algebra $X$ of Example 2.4. Then easy calculations give that (15) is satisfied but (12) is not satisfied because $(1 * 2) * 3 = 0 * 1 = ((1 * 3) * 3) * (2 * 3)$. Further the following theorem tells us that BCH-algebras satisfying (12) are BCI-algebras.

**Theorem 3.4.** A BCH-algebra satisfying $(x * y) * z = ((x * z) * z) * (y * z)$ is a BCI-algebra.
**Proof.** In view of (11) it is sufficient to prove that (1) holds. Consider
\[
((x \ast y) \ast (x \ast z))\ast (z \ast y) = ((x \ast (x \ast z)) \ast y) \ast (z \ast y)
\]
= \(((x \ast y) \ast y) \ast ((x \ast z) \ast y)) \ast (z \ast y) \text{ (by (12))}
\]
= \(((x \ast y) \ast y) \ast (z \ast y)) \ast ((x \ast z) \ast y) \text{ (by (12))}
\]
= 0.

(3.1)

This completes the proof.

In view of Theorems 3.3 and 3.4 and the comments made between them, we adopt the following definitions for BCH-algebras.

**Definition 3.5.** A BCH-algebra $X$ is weakly positive implicative if
\[
x \ast y = ((x \ast y) \ast y) \ast (0 \ast y) \quad \forall x, y \in X.
\]

(3.2)

We note that the BCH-algebra of Example 2.4 satisfies (3.2). Thus there exist proper weakly positive implicative BCH-algebras.

**Definition 3.6.** A BCH-algebra $X$ is weakly implicative if
\[
(x \ast (y \ast x)) \ast (0 \ast (y \ast x)) = x \quad \forall x, y \in X.
\]

(3.3)

**Definition 3.7.** A BCH-algebra $X$ is weakly commutative if
\[
(x \ast (x \ast y)) \ast (0 \ast (x \ast y)) = y \ast (y \ast x).
\]

(3.4)

**Theorem 3.8.** Every weakly implicative BCH-algebra $X$ is a weakly positive implicative BCH-algebra.

**Proof.** Let $X$ be weakly implicative. Then
\[
(x \ast (z \ast x)) \ast (0 \ast (z \ast x)) = x.
\]

(3.5)

Putting $x = z \ast x$ in (3.5), we get
\[
((z \ast x) \ast (z \ast (z \ast x))) \ast (0 \ast (z \ast (z \ast x))) = z \ast x.
\]

(3.6)

Since $z \ast (z \ast x) \leq x$, therefore (10) gives $0 \ast (z \ast (z \ast x)) = 0 \ast x$. Thus
\[
z \ast x = ((z \ast x) \ast (z \ast (z \ast x))) \ast (0 \ast x) = ((z \ast (z \ast (z \ast x))) \ast x) \ast (0 \ast x).
\]

(3.7)

Now
\[
(z \ast x) \ast (z \ast (z \ast x)))
\]
= \(((z \ast (z \ast (z \ast x))) \ast x) \ast (0 \ast x)) \ast (z \ast (z \ast x))
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= \(((z \ast (z \ast (z \ast x))) \ast x) \ast (z \ast (z \ast (z \ast x)))) \ast (0 \ast x)
\]
= (0 \ast x) \ast (0 \ast x) = 0.

(3.8)
Hence $z \star x \leq z \star (z \star (z \star x)) \leq z \star x$. Thus

$$z \star (z \star (z \star x)) = z \star x \quad (3.9)$$

holds in a weakly implicative BCH-algebra. Putting (3.9) in (3.7) we get $z \star x = ((z \star x) \star x) \star (0 \star x)$. Hence, $X$ is weakly positive implicative. This completes the proof.

**Remark 3.9.** It is known that $0 \star x = 0 \star (0 \star (0 \star x))$ holds in a BCH-algebra [3], but it is still not known that in a BCH-algebra the identity $x \star y = x \star (x \star (x \star y))$ holds or not, although it holds in BCI-algebras and weakly implicative BCH-algebras (as shown in (3.9)).

**Remark 3.10.** Since every BCI-algebra is a BCH-algebra and weak positive implicativeness and weak implicativeness coincide with positive implicativeness and implicativeness, respectively, in BCK-algebras [1], therefore the following results of Chaudhry and Iséki follow as corollaries of Theorem 3.4.

**Corollary 3.11** (see [1]). Every weakly implicative BCI-algebra is a weakly positive implicative BCI-algebra.

**Corollary 3.12** (see [8]). Every implicative BCK-algebra is a positive implicative BCK-algebra.

**Theorem 3.13.** A BCH-algebra $X$ satisfying $(x \star (x \star y)) \star (0 \star (x \star y)) = y \star (y \star x)$ is a BCI-algebra.

**Proof.** It is sufficient to show that (1) holds. We consider

$$((x \star y) \star (x \star z)) \star (z \star y)$$

$$= (((x \star (x \star z)) \star y) \star (z \star y)$$

$$= (((z \star (z \star x)) \star (0 \star (z \star x))) \star (z \star y)$$

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$$= (0 \star (z \star x)) \star (0 \star (z \star x)) = 0. \quad (3.10)$$

This completes the proof.

We now pose the following open problem.

**Open Problem 1.** Do there exist classes of proper BCH-algebras other than the classes of weakly positive implicative and weakly implicative BCH-algebras, which are generalizations of the known classes of BCI- as well as BCK-algebras.

4. Characterization of weakly positive implicative BCH-algebras. In this section, we characterize weakly positive implicative BCH-algebras by their self maps.

**Definition 4.1.** Let $X$ be a BCH-algebra. For a fixed $x$ in $X$, the map $R_x : X \to X$ given by $R_x(t) = t \star x$ for all $t \in X$ is called a right self map.
**Definition 4.2.** Let $X$ be a BCH-algebra. For a fixed $x$ in $X$, the map $R'_x : X \rightarrow X$ given by $R'_x(t) = (t \ast x) \ast (0 \ast x)$ for all $t \in X$ is called a weak right self map.

The following theorem gives us a characterization of a weakly positive implicative BCH-algebra with the help of its right and weak right self maps.

**Theorem 4.3.** A BCH-algebra $X$ is weakly positive implicative if and only if $R_z = R'_z \circ R_z$ for all $z \in X$, where "$\circ$" is composition of functions.

**Proof.** Let $X$ be a BCH-algebra and $R_z = R'_z \circ R_z$. Then $R_z(y) = R'_z \circ R_z(y)$ for all $y \in X$. Thus $y \ast z = R'_z(y \ast z) = ((y \ast z) \ast z) \ast (0 \ast z)$ for all $y, z \in X$. Hence $X$ is a weakly positive implicative BCH-algebra. Conversely, if $X$ is a weakly positive implicative BCH-algebra, then $y \ast z = ((y \ast z) \ast z) \ast (0 \ast z)$. Thus $R_z(y) = (R_z(y) \ast z) \ast (0 \ast z) = R'_z(R_z(y)) = R'_z \circ R_z(y)$ for all $y, z \in X$. Hence $R_z = R'_z \circ R_z$. This completes the proof.

**Theorem 4.4.** Let $X$ be a weakly positive implicative BCH-algebra. Then $R'_{xy} = R'_{x} \circ R'_{y} = (R'_{xy})^2$.

**Proof.** Since $X$ is weakly positive implicative, therefore $x \ast y = ((x \ast y) \ast y) \ast (0 \ast y)$. Thus

\[
(x \ast y) \ast (0 \ast y) = (((x \ast y) \ast y) \ast (0 \ast y)) \ast (0 \ast y) = (x \ast y) \ast (0 \ast y).
\]

Hence

\[
R'_{xy}(x) = R'_{xy}((x \ast y) \ast (0 \ast y)) = R'_{xy}(R'_{xy}(x)) = R'_{xy} \circ R'_{xy}(x) = (R'_{xy})^2(x)
\]

for all $x, y \in X$. This completes the proof.

The following example shows that the converse of the above theorem is not true.

**Example 4.5.** Let $X = \{0, a, b, c\}$ in which $*$ is defined by:

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Then $X$ is a BCI-algebra. Further $X$ is not weakly positive implicative because $a = c \ast b \ast ((c \ast b) \ast b) \ast (0 \ast b) = (a \ast b) \ast (0 \ast b) = b \ast b = 0$. Moreover, easy calculations give that

\[
R'_0 = (R'_0)^2, \quad R'_a = (R'_a)^2, \quad R'_b = (R'_b)^2, \quad R'_c = (R'_c)^2.
\]

This shows that the converse of Theorem 4.4 does not hold for the class of BCH-algebras, because it does not hold for BCI-algebras.

We now pose another open problem.
Open problem 2. What are the characterizations of weakly positive implicative BCH-algebras and weakly implicative BCH-algebras in terms of their ideals.

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References


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