PARTIAL SUMS OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. The object of the present paper is to consider the starlikeness and convexity of partial sums of certain analytic functions in the open unit disk.

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1. Introduction. Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Let $S^*(\alpha)$ be the subclass of $A$ consisting of functions $f(z)$ which satisfy

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$. A function $f(z)$ in $S^*(\alpha)$ is said to be starlike of order $\alpha$ in $U$. Furthermore, let $K(\alpha)$ denote the subclass of $A$ consisting of all functions $f(z)$ which satisfy

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$. A function $f(z)$ belonging to $K(\alpha)$ is said to be convex of order $\alpha$ in $U$. We note that $f(z) \in S^*(\alpha)$ if and only if $zf'(z) \in K(\alpha)$ and denote by $S^*(0) \equiv S^*$ and $K(0) \equiv K$. For $f(z) \in A$, we introduce the partial sum of $f(z)$ by

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k.$$  (1.4)

Remark 1.1. It is well known that

(i) $f(z) = z/(1-z)^2 = z + \sum_{k=2}^{\infty} k z^k$ is the extremal function for the class $S^*$. But $f_2(z) = z + 2z^2 \in S^*$.

(ii) $f(z) = z/(1-z) = z + \sum_{k=2}^{\infty} z^k$ is the extremal function for the class $K$. But $f_2(z) = z + z^2 \notin K$.

For the partial sums $f_n(z)$ of $f(z) \in S^*$, Szegö [2] showed the following theorem.
Theorem 1.2. (i) $f(z) \in S^*$ implies that $f_n(z) \in S^*$ for $|z| < 1/4$. The result is sharp.
(ii) $f(z) \in S^*$ implies that $f_n(z) \in K$ for $|z| < 1/8$. The result is sharp.

Further, Padmanabhan [1] proved the following theorem.

Theorem 1.3. If $f(z)$ is 2-valently starlike in $U$, then $f_n(z)$ is 2-valently starlike for $|z| < 1/6$. The result is sharp.

2. Function $F_n(z)$. We define the function $F_n(z)$ which is the partial sum of $f(z) \in A$ by

$$F_n(z) = z + a_n z^n.$$  \hfill (2.1)

Theorem 2.1. The function $F_n(z)$ satisfies

$$\frac{1 - n \frac{\alpha}{a_n} r^{n-1}}{1 - \frac{\alpha}{a_n} r^{n-1}} \leq \text{Re} \left[ \frac{z F_n'(z)}{F_n(z)} \right] \leq \frac{1 + n \frac{\alpha}{a_n} r^{n-1}}{1 + \frac{\alpha}{a_n} r^{n-1}} \tag{2.2}$$

for $0 \leq r < \frac{1}{n|a_n|}$. Therefore, $F_n(z) \in S^*(\alpha)$ for $0 \leq r < \frac{1}{n|a_n|}$.

Proof. Note that

$$\frac{z F_n'(z)}{F_n(z)} = \frac{z + n a_n z^n}{z + a_n z^n} = n - \frac{n-1}{1 + a_n z^{n-1}}. \tag{2.3}$$

It follows from (2.3) that

$$\text{Re} \left[ \frac{z F_n'(z)}{F_n(z)} \right] = n - (n-1) \frac{1 + |a_n| r^{n-1} \cos \theta}{1 + |a_n|^2 r^{2(n-1)} + 2 |a_n| r^{n-1} \cos \theta}. \tag{2.4}$$

Since, the right-hand side of (2.4) is increasing for $\cos \theta$ if $|a_n| < 1$, we obtain (2.2). Further, we also see that

$$\text{Re} \left[ \frac{z F_n'(z)}{F_n(z)} \right] \geq \frac{1 - n |a_n| r^{n-1}}{1 - |a_n| r^{n-1}} > \alpha \tag{2.5}$$

for $0 \leq r < \frac{1}{n|a_n|}$. This completes the proof of the theorem.

Next, we derive the following theorem.

Theorem 2.2. The function $F_n(z)$ satisfies

$$\frac{1 - n^2 |a_n| r^{n-1}}{1 - n |a_n| r^{n-1}} \leq \text{Re} \left[ 1 + \frac{z F_n''(z)}{F_n(z)} \right] \leq \frac{1 + n^2 |a_n| r^{n-1}}{1 + n |a_n| r^{n-1}} \tag{2.6}$$

for $0 \leq r < \frac{1}{n |a_n|}$. Therefore, $F_n(z) \in K$ for $0 \leq r < \frac{1}{n |a_n|}$.

Proof. Noting that

$$1 + \frac{z F_n''(z)}{F_n(z)} = n - \frac{n-1}{1 + n a_n z^{n-1}}, \tag{2.7}$$

we have

$$\text{Re} \left[ 1 + \frac{z F_n''(z)}{F_n(z)} \right] = n - (n-1) \frac{1 + n |a_n| r^{n-1} \cos \theta}{1 + n^2 |a_n|^2 r^{2(n-1)} + 2 n |a_n| r^{n-1} \cos \theta}, \tag{2.8}$$

which derives (2.6).
By virtue of Theorems 2.1 and 2.2, we have the following conjecture.

**Conjecture 2.3.** For the partial sum \( f_n(z) \) of \( f(z) \) belonging to the class \( A \),

(i) \( f_n(z) \in S^*(\alpha) \) for \( 0 \leq r < \frac{1}{n(1-\alpha)}|a_n| \leq 1 \),

(ii) \( f_n(z) \in K(\alpha) \) for \( 0 \leq r < \frac{1}{n(1-\alpha)}|a_n| \leq 1 \).

3. The partial sums of certain analytic functions. In this section, we consider the partial sums of functions \( f(z) = z/(1-z) \) and \( f(z) = z/(1-z)^2 \).

**Theorem 3.1.** Let \( f_3(z) = z/z^2 + z^3 \) be the partial sum of \( f(z) = z/(1-z) \) which is the extremal function of the class \( K \). Then \( f_3(z) \in S^*(626/961) \) for \( 0 \leq r < \beta (1/7 < \beta < 1/6) \), where \( \beta \) is the positive root of

\[
x^4 - 8x^3 + 9x^2 - 8x + 1 = 0 \quad (0 < x < \frac{1}{\sqrt{3}}).
\]

**Proof.** We consider \( \alpha \) such that

\[
\text{Re} \left[ \frac{z f_3'(z)}{f_3(z)} \right] = \text{Re} \left[ 3 - \frac{2 + z}{1 + z^2 + z^3} \right] > \alpha
\]

for \( 0 < r < \beta \). This implies that

\[
\text{Re} \left[ \frac{2 + z}{1 + z^2 + z^3} \right] = 1 + \frac{(1-r^2)(1+r^2+r \cos \theta)}{1-r^2+r^4+4r^2 \cos^2 \theta + 2r(1+r^2) \cos \theta} < 3 - \alpha,
\]

that is,

\[
\text{Re} \left[ \frac{(1-r^2)(1+r^2+r \cos \theta)}{1-r^2+r^4+4r^2 \cos^2 \theta + 2r(1+r^2) \cos \theta} \right] < 2 - \alpha.
\]

Let the function \( g(t) \) be given by

\[
g(t) = \frac{(1-r^2)(1+r^2+rt)}{1-r^2+r^4+4r^2t^2+2r(1+r^2)t} \quad (t = \cos \theta).
\]

Then, we have

\[
g'(t) = \frac{r(r+1)(r-1)(1+5r^2+r^4+4r^2t^2+8r(1+r^2)t)}{(1-r^2+r^4+4r^2t^2+2r(1+r^2)t)^2}.\]

Letting

\[
h(t) = 1+5r^2+r^4+4r^2t^2+8r(1+r^2)t,
\]

we see that (i) \( h(t) < 0 \Rightarrow g'(t) > 0 \), (ii) \( h(t) > 0 \Rightarrow g'(t) < 0 \), and (iii) \( h(t) = 0 \) for \( t = (-2(1+r^2) \pm \sqrt{3(1+r^2+r^4)})/2r \).

If we write

\[
t_1 = \frac{-2(1+r^2)+\sqrt{3(1+r^2+r^4)}}{2r} < 0,
\]

then, \( 0 \leq r \leq \beta \) implies that \( t_1 \leq -1 \), so that, \( h(t) \geq 0 \). This gives us that

\[
g(t) \leq g(-1) = \frac{1-r+r^3-r^4}{1-2r+3r^2-2r^3+r^4} = \frac{g_1(r)}{g_2(r)}.
\]
It is easy to check that $g_1(r)$ is decreasing for $r$ ($0 \leq r < 1/\sqrt{3}$). Therefore,
\[ \frac{8 - 2\sqrt{3}}{9} = g_1\left(\frac{1}{\sqrt{3}}\right) < g_1(r) \leq g_1(0) = 1. \tag{3.10} \]

Also, $g_2(r)$ is decreasing for $r$ ($0 \leq r < \beta$), because $g'_2(0) = -2 < 0$ and $g'_2(1/6) = -31/27 < 0$. This gives that
\[ \frac{961}{1296} = g_2\left(\frac{1}{6}\right) < g_2(r) \leq g_2(0) = 1. \tag{3.11} \]

Consequently, we conclude that
\[ g(t) \leq g(-1) = \frac{g_1(r)}{g_2(r)} < \frac{1296}{961} = 2 - \alpha, \tag{3.12} \]
that is, $\alpha = 626/961 = 0.651 \ldots$. Thus, we have
\[ \text{Re}\left[ \frac{z f'_3(z)}{f_3(z)} \right] > \alpha \quad (\alpha = \frac{626}{961}) \tag{3.13} \]
for $0 \leq r < \beta$.

Finally, we obtain the following theorem.

**Theorem 3.2.** Let $f_3(z) = z + 2z^2 + 3z^3$ be the partial sum of the Koebe function $f(z) = z/(1 - z)^2$ which is the extremal function for the class $S^*$. Then $f_3(z) \in K(3191/15876)$ for $0 \leq r < \beta$ ($1/14 < \beta < 113$), where $\beta$ is the positive root of
\[ 81x^4 - 162x^3 + 72x^2 - 18x + 1 = 0 \quad (0 \leq x < \frac{1}{3}). \tag{3.14} \]

**Proof.** Since
\[ \text{Re}\left[ 1 + \frac{zf''_3(z)}{f_3(z)} \right] = \text{Re}\left[ 3 - \frac{2(1+2z)}{1+4z+9z^2} \right] > \alpha \tag{3.15} \]
implies that
\[ \text{Re}\left[ \frac{1+2z}{1+4z+9z^2} \right] = \frac{1}{2} + \frac{4r(1-r^2)\cos \theta + 1 - 81r^4}{2(1 - 2r^2 + 81r^4 + 8r(1 + 9r^2)\cos \theta + 36r^2\cos^2 \theta)} < \frac{3 - \alpha}{2}, \tag{3.16} \]
we have to check that
\[ \frac{(1-9r^2)(1+9r^2+4r\cos \theta)}{1-2r^2+81r^4+8r(1 + 9r^2)\cos \theta + 36r^2\cos^2 \theta} < 2 - \alpha. \tag{3.17} \]
If we let
\[ h(t) = \frac{(1-9r^2)(1+9r^2+4rt)}{1-2r^2+81r^4+8r(1 + 9r^2)t + 36r^2t^2}, \tag{3.18} \]
then, we have
\[ h(t) \leq h(-1) = \frac{1-4r+36r^3-81r^4}{1-8r+34r^2-72r^3+81r^4} = \frac{g_1(r)}{g_2(r)}. \tag{3.19} \]
Noting that $0 < g_1(r) < 1$, and $g_2(r) > g_2(1/13) = 15876/28561$, we have

$$h(t) \leq h(-1) < \frac{1}{g_2(r)} < \frac{28561}{15876} = 2 - \alpha,$$

which implies that $\alpha = 3191/15876 = 0.200\ldots \square$

**References**


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