ON THE PROJECTIONS OF LAPLACIANS UNDER RIEMANNIAN SUBMERSIONS

HUILING LE

(Received 31 December 1998)

ABSTRACT. We give a condition on Riemannian submersions from a Riemannian manifold \( M \) to a Riemannian manifold \( N \) which will ensure that it induces a differential operator on \( N \) from the Laplace-Beltrami operator on \( M \). Equivalently, this condition ensures that a Riemannian submersion maps Brownian motion to a diffusion.

2000 Mathematics Subject Classification. Primary 58J65, 53C21.

1. Introduction. Suppose that \( M, N \) are, respectively, \( m \)- and \( n \)-dimensional Riemannian manifolds and that \( m > n \). Both \( M \) and \( N \) will then carry Laplace-Beltrami operators \( \triangle_M \) and \( \triangle_N \), respectively, determined by the Riemannian metrics.

Let the mapping \( \pi : M \rightarrow N \) such that \( \pi(\sigma_m) = \sigma_n \) be a Riemannian submersion. Normally, the Laplace-Beltrami operator \( \triangle_M \) will not induce a differential operator on \( N \) under the submersion \( \pi \) because \( \triangle_M \) may depend not only on \( \pi(\sigma_m) \) but also on \( \sigma_m \). Equivalently, a Brownian motion on \( M \) will not normally be mapped by \( \pi \) to a diffusion on \( N \) because it may happen that our prediction of \( \sigma_n(t + u) \) \((u > 0)\) will be improved if we know where \( \sigma_m(t) \) lies in \( \pi^{-1}(\sigma_n(t)) \), and we can expect to get information about \( \sigma_n(t) \) from the past history \( \{\sigma_n(t - v) : 0 \leq v < t\} \) of the submersed process. However, once we know that there is a differential operator \( \mathcal{L} \) on \( N \) that satisfies the relation

\[
(\mathcal{L}\phi) \circ \pi = \triangle_M(\phi \circ \pi),
\]

we can find several equivalent expressions for \( \mathcal{L} \) in terms of the volume, the second fundamental form, and the mean curvature of the fibres, respectively, which will be listed here.

(a) If the fibres are compact, let \( v(\sigma_n) \) be the \((m - n)\)-dimensional volume of the fibre \( \pi^{-1}(\sigma_n) \) and \( V \) the vector field \( \text{grad} \log v \). Carne’s formula (cf. [3]) then tells us that

\[
\mathcal{L}\phi = \triangle_N \phi + V \phi = \triangle_N \phi + \langle V, \text{grad} \phi \rangle.
\]

(b) Recall that \( \triangle_M \) can be written in terms of any given orthonormal vector fields \( X_1, \ldots, X_m \) on \( M \) as

\[
\triangle_M = \sum_{i=1}^{m} \{X_iX_i - \nabla X_i X_i\},
\]

the operator \( \nabla \) here being the Levi-Civita connection. Therefore, we choose \( Y_1, \ldots, Y_n \) to be orthonormal vector fields in a neighborhood of \( \sigma_n \in N \), \( X_1, \ldots, X_m \) the unique
horizontal lifts of \( Y_1, \ldots, Y_n \) to a neighborhood of \( \sigma_m \in \pi^{-1}(\sigma_n) \) (so that \( X_1, \ldots, X_n \) are orthonormal vector fields on the \( \pi \)-related horizontal subspace of \( \mathcal{T}(M) \)) and then supplement the latter by \( m - n \) orthonormal vertical vector fields \( X_{n+1}, \ldots, X_m \) in the same neighborhood. \( \Delta_M \) at \( \sigma_m \) can thus be written as

\[
\Delta_M = \sum_{i=1}^{n} \{ X_iX_i - \nabla_{X_i}X_i \} + \sum_{i=n+1}^{m} \{ X_iX_i - \nabla_{X_i}X_i \}. \tag{1.4}
\]

However, for any smooth function \( \phi : N \to \mathbb{R} \), the composed function \( \phi \circ \pi : M \to \mathbb{R} \) will be constant along each fibre \( \pi^{-1}(\sigma_n) \), and hence

\[
X_i(\phi \circ \pi) = \begin{cases} 
(Y_i\phi) \circ \pi, & 1 \leq i \leq n, \\
0, & n + 1 \leq i \leq m. 
\end{cases} \tag{1.5}
\]

And, on the other hand, \( \nabla_{X_i}X_i \) is equal to the sum of the horizontal lift of \( \nabla_{Y_i}Y_i \) and \( \nabla_{V_i} \), where each \( \nabla_{V_i} \) is the vertical component of \( \nabla_{X_i}X_i \). Thus

\[
\Delta_M(\phi \circ \pi) = (\Delta_N \phi) \circ \pi
- \sum_{i=n+1}^{m} \left[ \text{the } \pi \text{-related horizontal component of } \nabla_{X_i}X_i \right] (\phi \circ \pi). \tag{1.6}
\]

The Hessian of a function \( \phi \) is the symmetric \((0,2)\) tensor field defined by

\[
\text{Hess}(\phi)(X,Y) = XY\phi - (\nabla_X Y)\phi, \tag{1.7}
\]

and the so-called shape tensor (or "second fundamental form" tensor) of each fibre \( \pi^{-1}(\sigma_n) \) is the bilinear symmetric mapping \( \Pi \) from \( \mathcal{X}(\pi^{-1}(\sigma_n)) \times \mathcal{X}(\pi^{-1}(\sigma_n)) \) to \( \mathcal{X}(\pi^{-1}(\sigma_n))^\perp \), where \( \mathcal{X}(\pi^{-1}(\sigma_n)) \) denotes the set of all smooth vertical vector fields of \( M \) defined on \( \pi^{-1}(\sigma_n) \), such that \( \Pi(X_1,X_2) \) is the component of \( \nabla_{X_1}X_2 \) in \( \mathcal{T}(M) \) normal to the fibre \( \pi^{-1}(\sigma_n) \). It turns out that

\[
\text{Hess}(\phi \circ \pi)(X_i,X_i) = -\nabla_{X_i}X_i(\phi \circ \pi)
= -\langle \Pi(X_i,X_i), \text{grad}(\phi \circ \pi) \rangle, \quad n + 1 \leq i \leq m, \tag{1.8}
\]

and so an equivalent expression for \( \mathcal{L} \) is

\[
(\mathcal{L}\phi) \circ \pi = (\Delta_N \phi) \circ \pi + \sum_{i=n+1}^{m} \text{Hess}(\phi \circ \pi)(X_i,X_i)
= (\Delta_N \phi) \circ \pi - \left\langle \sum_{i=n+1}^{m} \Pi(X_i,X_i), \text{grad}(\phi \circ \pi) \right\rangle. \tag{1.9}
\]

(c) Moreover, for any \((m-n)\)-dimensional submanifold \( M_0 \) of \( M \), the mean curvature vector field \( H_{M_0} \) of \( M_0 \) at \( p \in M_0 \) is given by

\[
H_{M_0}(p) = \frac{1}{m-n} \sum_{i=n+1}^{m} \Pi(E_i,E_i), \tag{1.10}
\]
where $E_{n+1}, \ldots, E_m$ is any orthonormal basis for the tangent space $\mathcal{T}_p(M_0)$. It is easy to check that if $x_{n+1}, \ldots, x_m$ is an adapted coordinate system for $M_0$, then

$$H_{M_0} = \frac{1}{m-n} \sum_{i,j=n+1}^m \theta_M^{ij} \Pi \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right),$$  \hspace{1cm} (1.11)

and that if $\partial/\partial x_1, \ldots, \partial/\partial x_n$ are normal to $M_0$, then

$$\Pi \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \sum_{r=1}^n (\Gamma_M)^i_{jr} \frac{\partial}{\partial x_r}, \quad n + 1 \leq i, j \leq m.$$  \hspace{1cm} (1.12)

It follows from (1.9) that

$$(\mathcal{L} \phi) \circ \pi = (\triangle_N \phi) \circ \pi - (m - n) \langle H_\pi^{-1}, \text{grad} (\phi \circ \pi) \rangle,$$  \hspace{1cm} (1.13)

which gives another expression for $\mathcal{L}$ when it exists.

So a problem rises here: what is the condition for such a differential operator $\mathcal{L}$ to exist, that is, when does the submersion $\pi$ map a Brownian motion on $M$ to a diffusion on $N$?

The above discussion shows that $\triangle_M (\phi \circ \pi) = (\mathcal{L} \phi) \circ \pi$ for some operator $\mathcal{L}$ on $N$ if and only if the traces of the second fundamental form for each fibre $\pi^{-1}(\sigma_n)$ are $\pi$-related on that fibre, or equivalently, if and only if the mean curvature vector fields $H_\pi^{-1}$ of each fibre $\pi^{-1}(\sigma_n)$ are $\pi$-related on that fibre, for evidently either of these is the necessary and sufficient condition that $\triangle_M$ depends only on $\pi(\sigma_m)$, and not on $\sigma_m$ itself.

We now discuss another condition in terms of the volume element of $M$ for the existence of $\mathcal{L}$.

**2. Some lemmas**

**Lemma 2.1.** Let $G_M$ and $G_N$ be the matrices of the local components of the metric tensor fields on $M$ and $N$ with respect to local coordinates $x: \sigma_m \rightarrow (x_1, \ldots, x_m)$ on $M$ and $y: \sigma_n \rightarrow (y_1, \ldots, y_n)$ on $N$, respectively, then

$$G_N^{-1} \circ \pi = JG_M^{-1} J^t,$$  \hspace{1cm} (2.1)

where $J$ is the Jacobian matrix of the coordinate representation $y \circ \pi \circ x^{-1}$ of $\pi$ with the $(i,j)$th entry

$$\frac{\partial (y_i \circ \pi)}{\partial x_j} = \frac{\partial (y_i \circ \pi \circ x^{-1})}{\partial x_j} \circ x,$$  \hspace{1cm} (2.2)

and $J^t$ is its transpose.

For any given local coordinate $y$ on $N$ at $\sigma_n$, there exists a local coordinate $x$ on $M$ at $\sigma_m \in \pi^{-1}(\sigma_n)$ such that

$$y \circ \pi \circ x^{-1}: (x_1, \ldots, x_m) = (y_1, \ldots, y_n, z_1, \ldots, z_{m-n}) \rightarrow (y_1, \ldots, y_n).$$  \hspace{1cm} (2.3)

This implies that $\pi$ is locally a fibration, that is, there exist a neighborhood $U_n$ of $\sigma_n \in N$, a neighborhood $U_m$ of $\sigma_m \in \pi^{-1}(\sigma_n)$, and a manifold $F$ such that $\pi^{-1}(U_n) \cap U_m$
is diffeomorphic to $U_n \times F$ and the diffeomorphism maps $\pi^{-1}(\sigma_n) \cap U_m$ to $F$, and that the $\pi$-related vertical subspace of $\mathcal{T}_{\sigma_m}(M)$ for $\sigma_m \in \pi^{-1}(\sigma_n)$ is spanned by $\partial/\partial x_{n+1}, \ldots, \partial/\partial x_m$. In general, however, $\partial/\partial x_1, \ldots, \partial/\partial x_n$ will not be horizontal to the $\pi$-related vertical subspace of $\mathcal{T}_{\sigma_m}(M)$.

$\pi$ is called integrable if the horizontal distribution, which is the orthogonal complement of $\text{Ker}(d\pi)$ in $\mathcal{T}(M)$, is integrable.

**Lemma 2.2.** $\pi$ is integrable, if and only if there exist local coordinates $x$ and $y$ satisfying the condition (2.3) for $M$ and $N$ such that the $\pi$-related horizontal subspace of $\mathcal{T}_{\sigma_m}(M)$ is spanned by $\partial/\partial x_1, \ldots, \partial/\partial x_n$.

**Proof.** If $\pi$-related horizontal subspace of $\mathcal{T}_{\sigma_m}(M)$ is spanned by $\partial/\partial x_1, \ldots, \partial/\partial x_n$, then the horizontal distribution, by definition, is integrable.

If $\pi$ is integrable, let $X_1, \ldots, X_n$ be the horizontal lifts of $\partial/\partial y_1, \ldots, \partial/\partial y_n$. Then the system of $n$ differential equations in $m$ variables

$$X_i f = 0, \quad 1 \leq i \leq n,$$

is complete. It follows that there are $m - n$ independent solutions $x_{n+1}, \ldots, x_m$ of (2.4), such that general solution of (2.4) is an arbitrary function of $x_{n+1}, \ldots, x_m$ (cf. [4]). Define $x_i = y_i \circ \pi$, for $1 \leq i \leq n$. Thus $x = (x_1, \ldots, x_m)$ is the coordinate we are looking for. In fact, for any given coordinate $y$ in $N$ we can always find a coordinate $\tilde{x}$ in $M$ such that (2.3) holds. Each $X_i$ can then be formulated as

$$X_i = \frac{\partial}{\partial \tilde{x}_i} + \sum_{j=1}^{m-n} \alpha_{ij} \frac{\partial}{\partial \tilde{x}_{j+n}},$$

where

$$(\alpha_{ij}) = E^{-1} F,$$

if the metric form of $M$ with respect to $\tilde{x}$ is

$$\tilde{G}_M = \begin{pmatrix} E_{n \times n} & F \\ F^t & G \end{pmatrix}^{-1}. $$

Thus, the metric form of $M$ with respect to $x$, by the fact that $X_i x_j = 0$, for $1 \leq i \leq n < j \leq m$, will be

$$G_M = \begin{pmatrix} E & 0 \\ 0 & H \end{pmatrix}^{-1},$$

for some positive definite symmetric matrix $H$. \hfill \Box

### 3. Main result

**Proposition 3.1.** If $\pi$ is integrable, then there is an operator $\mathcal{L}$ on $N$ with

$$(\mathcal{L} \phi) \circ \pi = \triangle_M (\phi \circ \pi),$$

if and only if the volume element $d\mu_M$ of $M$ can be expressed as a product of two independent forms: one is a composed $n$-form on $N$ with the submersion $\pi$ defined by

$$\{e^{(1/2)\phi} d\mu_N\} \circ \pi,$$
and the other is an \((m - n)\)-form on the fibres \(\pi^{-1}(\sigma_n)\), the local expression of which is denoted by
\[
\Psi^* d\sigma_{n+1} \cdots d\sigma_m, \tag{3.3}
\]
with the property that the latter will be independent of \(\sigma_n\) in a neighborhood of \(\sigma_n\). And when this condition is satisfied,
\[
\mathcal{L} = \Delta_N + \frac{1}{2} \nabla \Phi. \tag{3.4}
\]

**Proof.** The local form of the Laplace-Beltrami operator, in terms of any given coordinate \(x\) on \(M\), is
\[
\Delta_M = (\det G_M)^{-1/2} \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left( g_{M}^{ij} (\det G_M)^{1/2} \frac{\partial}{\partial x_j} \right). \tag{3.5}
\]
Thus for the coordinates \(x\) and \(y\) as Lemma 2.2, we are able to obtain that, for any smooth function \(\phi : N \to \mathbb{R}\),
\[
\Delta_M (\phi \circ \pi) = \sum_{i,j=1}^{m} \left\{ g_{M}^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial g_{M}^{ij}}{\partial x_j} \frac{\partial}{\partial x_i} + \frac{1}{2} g_{M}^{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial (\det G_M)}{\partial x_i} \right) \right\} (\phi \circ \pi)
+ \frac{1}{2} \sum_{i,j=1}^{m} \left\{ \frac{\partial}{\partial x_i} \left( \frac{\partial (\det G_M)}{\partial y_k} \right) \frac{\partial}{\partial y_l} \right\} (\phi \circ \pi)
+ \frac{1}{2} \sum_{i,j=1}^{m} \left\{ g_{M}^{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial (\det G_M)}{\partial y_k} \right) \frac{\partial}{\partial y_l} \right\} (\phi \circ \pi)
= \sum_{i,j=1}^{m} \left\{ g_{M}^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial g_{M}^{ij}}{\partial x_j} \frac{\partial}{\partial x_i} + \frac{1}{2} g_{M}^{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial (\det G_M)}{\partial x_i} \right) \right\} (\phi \circ \pi)
+ \frac{1}{2} \sum_{i,j=1}^{m} \left\{ \frac{\partial}{\partial x_i} \left( \frac{\partial (\det G_M)}{\partial y_k} \right) \frac{\partial}{\partial y_l} \right\} (\phi \circ \pi)
+ \frac{1}{2} \sum_{i,j=1}^{m} \left\{ g_{M}^{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial (\det G_M)}{\partial y_k} \right) \frac{\partial}{\partial y_l} \right\} (\phi \circ \pi)
= (\Delta_N \phi) \circ \pi - \sum_{i,j=1}^{m} \frac{\partial}{\partial y_k} \left( g_{N}^{ij} \frac{\partial}{\partial y_l} \right) (\phi \circ \pi)
+ \frac{1}{2} \sum_{i,j=1}^{m} \left\{ \frac{\partial}{\partial x_i} \left( \frac{\partial (\det G_M)}{\partial y_k} \right) \frac{\partial}{\partial y_l} \right\} (\phi \circ \pi)
+ \frac{1}{2} \sum_{i,j=1}^{m} \left\{ g_{M}^{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial (\det G_M)}{\partial y_k} \right) \frac{\partial}{\partial y_l} \right\} (\phi \circ \pi)
= (\Delta_N \phi) \circ \pi + \frac{1}{2} \sum_{j,k=1}^{m} g_{N}^{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial (\det G_M)}{\partial y_k} \right) \frac{\partial}{\partial y_l}. \tag{3.6}
\]
Note that here $\partial/\partial x_1, \ldots, \partial/\partial x_n$ are the horizontal lifts of $\partial/\partial y_1, \ldots, \partial/\partial y_n$. We know from the assumption that

$$\sigma_m \rightarrow \pi\text{-related horizontal subspace of } \mathcal{T}_{\sigma_m}(M)$$

is a distribution, and

$$\sum_{j=1}^{n} g_N^{ij} \circ \pi \frac{\partial}{\partial x_j}, \quad 1 \leq i \leq n,$$

forms a basis for it. Following the same discussion as in the proof of Lemma 2.2, we know that any solution of the system of differential equations

$$\sum_{j=1}^{n} g_N^{ij} \frac{\partial f}{\partial x_j} = 0, \quad 1 \leq i \leq n,$$

is a function of $x_{n+1}, \ldots, x_m$. On the other hand, we have by (1.2) that existence of $\mathcal{L}$ on $N$ if and only if there is a function $\Phi$ on $N$ such that

$$\sum_{j=1}^{n} g_N^{ij} \frac{\partial \Phi}{\partial y_j} (\log \left( \frac{\det G_M}{\det G_N \circ \pi} \right)) = \left\{ \sum_{j=1}^{n} g_N^{ij} \frac{\partial \Phi}{\partial y_j} \right\} \circ \pi, \quad 1 \leq i \leq n.$$ (3.10)

Therefore, the existence of $\mathcal{L}$ is equivalent to that there is a function $\Psi$ of $x_{n+1}, \ldots, x_m$ such that

$$\det G_M = e^{\Phi \circ \pi + \Psi(x_{n+1}, \ldots, x_m)} \det G_N \circ \pi.$$ (3.11)

$e^\Phi \det G_N$ is clearly a function on $N$. If we define a function $\Psi^*$ on a neighborhood of $\sigma_m \in \pi^{-1}(\sigma_n)$, as the restriction of the function $\sigma^{(1/2)} \Phi$ on $\pi^{-1}(\sigma_n)$, then $\Psi^*$ is independent on fibres in a neighborhood of $\sigma_m \in \pi^{-1}(\sigma_n)$, and $\det G_M$ is a product of a composed function on $N$ with $\pi$ and a function on the fibres of $\pi$.

The above discussion shows that the volume element $d\mu_M$ on $M$ is here expressed as

$$d\mu_M(\sigma_m) = \left( e^{(1/2)\Phi} \sqrt{\det G_N} \right) \circ \pi(\sigma_m) \, dx_1 \cdots dx_n(\sigma_m)$$

$$\times \Psi^*(\sigma_m) \, dx_{n+1} \cdots dx_m(\sigma_m)$$

$$= \left( e^{(1/2)\Phi} \sqrt{\det G_N} \right) (\sigma_n) \, dy_1 \cdots dy_n(\sigma_n)$$

$$\times \Psi^*(\sigma_m) \, dx_{n+1} \cdots dx_m(\sigma_m).$$ (3.12)

Because $\pi$ is a submersion, $M$ is locally diffeomorphic to $N \times F$ for a $(m - n)$-dimensional manifold $F$, and so the above condition is equivalent to that the volume element $d\mu_M$ can locally be expressed as a product of a composed $n$-form on $N$ with the submersion $\pi$ and an $(m - n)$-form on $F$. \qed
4. Remarks. (a) We know from the proof of Proposition 3.1 that, for any general coordinates such that (2.3) holds,

\[ \Delta_M(\phi \circ \pi) = (\Delta_N \phi) \circ \pi + \sum_{k=1}^{n} \left\{ \frac{1}{2} \sum_{j=1}^{m} g_{M}^{kj} \frac{\partial}{\partial x_j} \left( \log \left( \frac{\det G_M}{\det G_N \circ \pi} \right) \right) + \sum_{j=n+1}^{m} \frac{\partial g_{M}^{kj}}{\partial x_j} \left( \frac{\partial \phi}{\partial y_k} \circ \pi \right). \right\} \]

(4.1)

Compared with (1.6), we know that the \( k \)th \( (1 \leq k \leq n) \) component of the vector

\[ \sum_{i=n+1}^{m} \{ \text{the } \pi \text{-related horizontal component of } \nabla X_i X_i \} \]

(4.2)

is

\[ -\frac{1}{2} \sum_{j=1}^{m} g_{M}^{kj} \frac{\partial}{\partial x_j} \left( \log \left( \frac{\det G_M}{\det G_N \circ \pi} \right) \right) - \sum_{j=n+1}^{m} \frac{\partial g_{M}^{kj}}{\partial x_j}. \]

(4.3)

And compared with (1.2), we find that there is a differential operator \( \mathcal{L} \) on \( N \) with

\[ (\mathcal{L} \phi) \circ \pi = \Delta_M(\phi \circ \pi) \]

if and only if, for any \( 1 \leq k \leq n \), (4.3) is a function of \( \pi(\sigma_m) \), and

\[ \frac{1}{2} \sum_{j=1}^{m} g_{M}^{kj} \frac{\partial}{\partial x_j} \left( \log \left( \frac{\det G_M}{\det G_N \circ \pi} \right) \right) + \sum_{j=n+1}^{m} \frac{\partial g_{M}^{kj}}{\partial x_j} = \left\{ \sum_{j=1}^{n} g_{N}^{kj} \frac{\partial \log v}{\partial y_j} \right\} \circ \pi. \]

(4.4)

Therefore, for \( 1 \leq k \leq n \),

\[ \{ \text{grad}_N(\log v) \}_k \circ \pi = \frac{1}{2} \left\{ \text{grad}_M \log \frac{\det G_M}{\det G_N} \right\}_k + W_k, \]

(4.5)

where

\[ W_k = \sum_{j=n+1}^{m} \frac{\partial g_{M}^{kj}}{\partial x_j}, \]

(4.6)

that is, the first \( n \) components of \( \text{grad}_M((1/2)\log(\text{volume element of the fibre } \pi^{-1}(\sigma_m))) \) do not form a proper gradient of a function on \( N \), which usually depend not only on \( \pi(\sigma_m) \) but also on \( \sigma_m \).

When

\[ W_k = 0, \quad 1 \leq k \leq n. \]

(4.7)

Equation (4.4) can be rewritten as

\[ \sum_{j=1}^{m} g_{M}^{kj} \frac{\partial}{\partial x_j} \left\{ \log \left( \frac{\det G_M}{(v^2 \det G_N) \circ \pi} \right) \right\} = 0, \quad 1 \leq k \leq n, \]

(4.8)

that is equivalent to

\[ \left\langle dx_k, \sum_{j=1}^{m} \frac{\partial}{\partial x_j} \left\{ \log \left( \frac{\det G_M}{(v^2 \det G_N) \circ \pi} \right) \right\} dx_j \right\rangle = 0, \quad 1 \leq k \leq n, \]

(4.9)

that is,

\[ \left\langle dx_k, d \left\{ \log \left( \frac{\det G_M}{(v^2 \det G_N) \circ \pi} \right) \right\} \right\rangle = 0, \quad 1 \leq k \leq n, \]

(4.10)
so that
\[ d \left\{ \log \left( \frac{\det G_{M}}{(v^{2} \det G_{N} ) \circ \pi} \right) \right\} \]  
(4.11)
is orthogonal with all \( dx_{k} \) for \( 1 \leq k \leq n \) in \( \mathcal{T}^{\star}(M) \).

(b) When the condition in Proposition 3.1 holds, the volume element of the fibre \( \pi^{-1}(\sigma_{n}) \) is clearly
\[ e^{(1/2)\Phi(\sigma_{n})} \Psi^{\ast}(x_{n+1}, \ldots, x_{m}) \, dx_{n+1} \cdots dx_{m}; \]  
(4.12)
and so if \( \pi \) is also a fibration with compact fibre \( F \), the \( (m-n) \)-dimensional volume \( v(\sigma_{n}) \) of the fibre \( \pi^{-1}(\sigma_{n}) \) will then be equal to
\[ v(\sigma_{n}) = e^{(1/2)\Phi(\sigma_{n})} \int_{F} \Psi^{\ast}(x_{n+1}, \ldots, x_{m}) \, dx_{n+1} \cdots dx_{m} = \kappa e^{(1/2)\Phi(\sigma_{n})}, \]  
(4.13)
for some constant \( \kappa \), which coincides with (1.2).

(c) The condition of integrability of \( \pi \) in Proposition 3.1 should be able to be weakened. We study the following two cases.

(i) For the submersion \( \pi \) with minimal fibres, in particular with totally geodesic fibres, it is known that \( \mathcal{E} = \triangle_{N} \), which follows immediately from the fact that the term
\[ \sum_{i=n+1}^{m} \text{[the } \pi \text{-related horizontal component of } \nabla_{X_{i}}X_{i} \text{]} \]  
(4.14)
in (1.6) vanishes by the definition of minimal submanifold.

On the other hand, when \( M \) is complete and \( \pi \) with totally geodesic fibres, we can also obtain from the fact that \( (M, N, \pi) \) is a fibre bundle with the Lie group of isometries of the fibre as structure group (cf. [5] and below) that
\[ d\mu_{M} = d\mu_{N} \circ \pi \times \Psi^{\ast} \, dx_{n+1} \cdots dx_{m}, \]  
(4.15)
for a suitable coordinate \( (x_{n+1}, \ldots, x_{m}) \) on fibres.

In the case that \( \pi \) is with minimal fibres, it follows from the fact that the structure group of the bundle (which is a priori the group of diffeomorphisms of the fibre \( F \)) reduces to the group of volume preserving diffeomorphisms of \( F \) (cf. [1]) that the volume element of \( M \) is of the expression (4.15).

(ii) The case that the submersion \( \pi \) is a quotient mapping with respect to a Lie group \( G \) of isometries acting properly and freely on \( M \).

The fibre \( \pi^{-1}(\sigma_{n}) \) here inherits a Riemannian structure from that of \( M \), and the corresponding volume element \( d\mu_{\pi^{-1}(\sigma_{n})} \) of the fibre \( \pi^{-1}(\sigma_{n}) \) is invariant under \( G \) by the transitive action of \( G \) of isometries on the fibres. Under the identification \( \pi^{-1}(\sigma_{n}) = G \), the volume elements \( d\mu_{\pi^{-1}(\sigma_{n})} \) and \( dg \), the unique left-invariant volume element up to constants of \( G \), must, by the uniqueness, be proportional (cf. [2]). Hence there exists a function \( e^{(1/2)\Phi} \) on \( N \) such that
\[ d\mu_{\pi^{-1}(\sigma_{n})} = e^{(1/2)\Phi(\sigma_{n})} \, dg, \]  
(4.16)
and so
\[ d\mu_{M} = dg \{ e^{(1/2)\Phi} \, d\mu_{N} \} \circ \pi, \]  
(4.17)
which gives a form for the volume element on \( M \) coincident with our claim if we notice that here \( M \) is locally diffeomorphic to \( N \times G \).
REFERENCES


Huiling Le: School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, UK
E-mail address: huiling.le@nottingham.ac.uk