THE INITIAL BOUNDARY VALUE PROBLEM OF A MIXED-TYPED HEMIVARIATIONAL INEQUALITY

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Abstract. A mixed-typed differential inclusion with a weakly continuous nonlinear term and a nonmonotone discontinuous nonlinear multi-valued term is studied, and the existence and decay of solutions are established.

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1. Introduction. In the present paper, the following initial boundary value problem of a degenerate multi-valued hyperbolic-parabolic inequality will be considered:

\[
\ddot{u}(t) + A(t)(\dot{u})(t) + B(u)(t) + \varphi(u(x,t)) \ni f(t), \quad \text{a.e. } t \in [0, T],
\]

\[
u(x, t) = 0, \quad \text{a.e. } (x, t) \in \Sigma = \partial \Omega \times [0, T],
\]

\[
u(0) = u_0, \quad \dot{u}(0) = u_1,
\]

where \(A\) is a weakly continuous operator; \(B\) is a linear, continuous, and symmetric operator; \(\varphi\) is a nonmonotone, discontinuous, and nonlinear set-valued mapping.

Physical motivations for studying inequality (1.1) come partly from problems of continuum mechanics and optimal control problems, where nonmonotone, nonlinear, discontinuous, and multi-valued constitutive laws and boundary or external constraints lead to various typed hemivariational inequalities, the mixed hyperbolic-parabolic hemivariational inequality is one of those [11, 12, 14].

For inequality (1.1), its stationary problems have been studied by many researchers (see [1, 2, 4, 13, 14, 15] and references therein). When \(\varphi\) degenerates into a class of single-valued mappings, inequality (1.1) becomes an equation, and when \(A\) and \(B\) were some special linear mappings and satisfy some conditions, equation (2.1) and some of its evolution equations have been investigated and applied intensively (see [5, 3, 6, 7, 8, 9, 10] and the references therein).

In this paper, we investigate the existence and decay of weak solution of the mixed hyperbolic-parabolic inequality (1.1) with \(\varphi, A,\) and \(B\) satisfying some conditions. We apply the Faedo-Galerkin method for the proof of existence of solutions.

2. Preliminaries. Let \(\Omega\) be a bounded open set of \(\mathbb{R}^n\) with regular boundary \(\Gamma\). Let \(T\) denote a positive real number, \(Q = \Omega \times [0, T]\). Suppose that \(b \in L^\infty_{\text{loc}}(\mathbb{R})\), for every \(\rho > 0\), set

\[
b_\rho(\xi) = \inf_{|\xi_1 - \xi| < \rho} b(\xi_1), \quad \overline{b}_\rho(\xi) = \sup_{|\xi_1 - \xi| < \rho} b(\xi_1),
\]

(2.1)
they are all monotone for $\rho > 0$. Set

\[ b_{\rho}(\xi) = \lim_{\rho \to 0^+} b_{\rho}(\xi), \quad \bar{b}(\xi) = \lim_{\rho \to 0^+} \bar{b}_{\rho}(\xi), \quad \varphi(\xi) = [b(\xi), \bar{b}(\xi)]. \tag{2.2} \]

Let $J(\xi) = \int_0^\xi b(t) \, dt$, then $\partial^C J(\xi) \subseteq \varphi(\xi)$, where $\partial^C J(\xi)$ denotes the Clarke-subdifferential of $J$ (see [2]). If $b(\xi_x)$ exists for every $\xi \in \mathbb{R}$, then $\varphi(\xi) = \partial^C J(\xi)$. If $b$ is continuous at the point $\xi$, then $\varphi(\xi)$ is single-valued at $\xi$, if $J$ is convex, $\varphi(\xi)$ is maximal monotone (see [2]).

Let $V = H^1_0(\Omega)$, $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$, $\langle \cdot, \cdot \rangle$ denotes the dual pair between $V$ and $V' = H^{-1}(\Omega)$ which is compatible with the inner product of $L^2(\Omega)$. Let $|x|_X$ denote the norm of the element $x$ of the Banach space $X$.

Considering the following initial boundary value problem of a hyperbolic-parabolic hemivariational inequality:

\[
\ddot{u}(t) + A(t)\dot{u}(t) + Bu(t) + g(t) = f(t), \quad \text{a.e.} \ t \in [0,T], \\
u(x,t) = 0, \quad \text{a.e.} \ (x,t) \in \Sigma = \partial \Omega \times [0,T], \\
u(0) = u_0, \quad \dot{u}(0) = u_1, \tag{2.3}
\]

\[
g(x,t) \in \varphi(u(x,t)), \quad \text{a.e.} \ (x,t) \in Q_T = \Omega \times [0,T],
\]

where $f$, $u_0$, and $u_1$ are given.

First we list some assumptions:

(1) $\exists c > 0, |b(\xi)| \leq c(1 + |\xi|), \text{a.e.} \ \xi \in \mathbb{R}$.

(2) $A : L^2(0,T; L^2(\Omega)) \to L^2(0,T; L^2(\Omega))$ is weakly continuous, and $A(t)$ is nonnegative, that is, $\langle A(t) v, v \rangle \geq 0$, for a.e. $t \geq 0$ and every $v \in L^2(\Omega)$.

(3) The function $t \to \langle A(t) u, v \rangle$ is measurable on $[0,T]$ for all $u, v \in L^2(\Omega)$.

(4) $B : H^1_0(\Omega) \to H^{-1}(\Omega)$ is linear, continuous, symmetric, and semicoercive, that is, $\exists c_1 > 0, c_2 > 0, c_3 > 0$

\[
|Bu|_{H^{-1}(\Omega)} \leq c_1 |v|_{H^1_0(\Omega)}, \quad \langle Bu, v \rangle = \langle Bv, u \rangle, \quad \forall u, v \in H^1_0(\Omega), \\
\langle Bv, v \rangle + c_3 |v|_{L^2(\Omega)}^2 \geq c_2 |v|_{H^1_0(\Omega)}^2, \quad \forall v \in H^1_0(\Omega). \tag{2.4}
\]

Let $\beta$ be any mollifier satisfying $\beta \in C^\infty(\mathbb{R})$, $\beta \geq 0$, supp $\beta \subset (-1,1)$, and $\int_{\mathbb{R}} \beta(\xi) \, d\xi = 1$. Set

\[
b_\varepsilon(\xi) = \frac{1}{\varepsilon} \int_{\mathbb{R}} \beta \left( \frac{\xi - z}{\varepsilon} \right) b(z) \, dz = \int_{|z| < 1} \beta(z) b(\xi - \varepsilon z) \, dz, \quad \text{for every } \varepsilon > 0. \tag{2.5}
\]

It is easy to see that $b_\varepsilon$ is a smooth function, and also satisfies assumption (1) with possible different constant $c$ if $b$ is agreeable with assumption (1). For convenience, we denote $b_{1/n}$ by $b_n$ for any positive integer $n$. 

3. Existence of solution

**Theorem 3.1.** Assume that \( f \in L^2(0,T;L^2(\Omega)) \), \( u_0 \in H^1_0(\Omega) \cap L^{p+1}(\Omega) \), \( u_1 \in L^2(\Omega) \). Then, under assumptions (1), (2), (3), and (4), there exists a function \( u \) defined in \( \Omega \times [0,T] \) such that

\[
\begin{align*}
    u &\in L^\infty(0,T;H^1_0(\Omega)) \cap C([0,T];L^2(\Omega)), \\
    \dot{u} &\in L^\infty(0,T;L^2(\Omega)) \cap C([0,T];H^{-1}(\Omega)), \\
    \ddot{u} &\in L^2(0,T;H^{-1}(\Omega)),
\end{align*}
\]

and

\[
\begin{align*}
    \dot{u}(t) + A(t)\dot{u}(t) + Bu(t) + g(t) &= f(t), \quad \text{in } L^2(0,T;H^{-1}(\Omega)), \\
    g(t) &\in \varphi(u(x,t)), \quad \text{a.e. } (x,t) \in \Omega \times [0,T], \\
    u(0) &= u_0, \quad \ddot{u}(0) = u_1.
\end{align*}
\]

**Proof.** Let \( \{e_n\}_{n=1}^\infty \) be a subset of \( V = H^1_0(\Omega) \) satisfying \( \text{span}\{e_n\} = V \), \( (e_i,e_j) = \delta_{ij} \). Let \( x_n = \sum_n \omega^1_i e_i - u_0 \) strongly in \( V \) and \( L^{p+1}(\Omega) \), \( y_n = \sum_n \omega^2_i e_i - u_1 \) strongly in \( L^2(\Omega) \).

Considering the following regularized equation of inequality (1.1)

\[
\xi^n = M^n + N^n + h, \quad \xi^n \big|_{t=0} = \omega^{1n}, \quad \xi^n \big|_{t=0} = \omega^{2n},
\]

where \( \xi^n = \{\xi^n_i\}_{1 \times n} \), \( \omega^{1n} = \{\omega^1_i\}_{1 \times n} \), \( \omega^{2n} = \{\omega^2_i\}_{1 \times n} \), \( h = \{(f,e_i)\}_{1 \times n} \), \( M^n = \{M^n_i\}_{1 \times n} \), \( M^n_i = -\langle A(t)\sum_n \xi^n_i e_j, e_i \rangle \), \( N^n = \{N^n_i\}_{1 \times n} \), \( N^n_i = -\langle B(\sum_n \xi^n_i e_j), e_i \rangle - \langle b_n(\sum_n \xi^n_j e_j), e_i \rangle \), where \( n \) denotes time derivate.

Equation (3.3) is a vector-valued ordinary differential equation and its local solution \( \xi^n \) exists on \( I_n = [0,T_n] \), \( 0 < T_n \leq T \). Set \( u_n(t) = \sum_n \xi^n_j e_j \) (\( t \in I_n \)). Equation (3.3) is equal to

\[
\langle \ddot{u}_n, e_i \rangle = -\langle A(t)\dot{u}_n, e_i \rangle - \langle Bu_n, e_i \rangle - \langle b_n(\dot{u}_n), e_i \rangle + \langle f, e_i \rangle, \quad i = 1,2,\ldots,n.
\]

Multiplying (3.4) by \( \xi^n_i \), summing over from \( i = 1 \) to \( i = n \) and integrating over \( [0,t] \) \( (t \leq T_n) \), we get

\[
\begin{align*}
|\ddot{u}_n(t)|^2_{L^2(\Omega)} + \langle Bu_n(t), u_n(t) \rangle + 2\int_0^t \langle A\ddot{u}_n, \dot{u}_n \rangle \, dt + 2\int_0^t \langle b_n(\dot{u}_n), \dot{u}_n \rangle \, dt \\
= 2\int_0^t \langle f, \dot{u}_n \rangle \, dt + \langle y_n, y_n \rangle + \langle Bx_n, x_n \rangle.
\end{align*}
\]
Using Gronwall’s inequality it follows that

\[
\int_0^t \langle b_n(u_n), \dot{u}_n \rangle \, d\tau = \int_\Omega \left\{ \int_0^{u_n(x,\tau)} b_n(\lambda) \, d\lambda \right\} \, dx,
\]

\[
= \int_\Omega \left\{ \int_0^{u_n(x,0)} b_n(\lambda) \, d\lambda - \int_0^{u_n(x,\tau)} b_n(\lambda) \, d\lambda \right\} \, dx,
\]

\[
\left| \int_0^t \langle b_n(u_n), \dot{u}_n \rangle \, d\tau \right| \leq c \int_\Omega \left\{ \left| u_n(x,\tau) \right| + \left| u_n(x,0) \right| + \int_0^{u_n(x,\tau)} \left| \lambda \right| \, d\lambda \right\} \, dx,
\]

\[
+ \left\{ \int_0^{u_n(x,0)} \left| \lambda \right| \, d\lambda \right\} \right\} \, dx,
\]

\[
\left(3.6\right)
\]

where \(|\Omega|\) denotes the Lebesgue measure of the domain \(\Omega\).

From \(3.5\), it follows that there exists \(c_4 > 0\) such that

\[
\left| \dot{u}_n(t) \right|_{L^2(\Omega)}^2 + \left| u_n(t) \right|_{H^1_0(\Omega)}^2 \leq c_4 + \left\{ c_3 + \frac{c}{2}(1 + |\Omega|) \right\} \left| u_n(t) \right|_{L^2(\Omega)}^2 + 2 \int_0^t \langle f, \dot{u}_n \rangle \, d\tau.
\]

\[
\left(3.7\right)
\]

We note that

\[
u_n(t) = u_n(0) + \int_0^t \dot{u}_n \, d\tau,
\]

\[
\left| u_n(t) \right|_{L^2(\Omega)}^2 \leq \left| u_n(0) \right|_{L^2(\Omega)}^2 + \int_0^t \left| u_n \right|_{L^2(\Omega)}^2 \, d\tau,
\]

using Hölder’s inequality, we get that there exists \(c_5, c_6 > 0\) such that

\[
\left| u_n(t) \right|_{L^2(\Omega)}^2 \leq c_5 + c_6 \int_0^t \left| \dot{u}_n \right|_{L^2(\Omega)}^2 \, d\tau,
\]

\[
\left(3.9\right)
\]

\[
\int_0^t \langle f, \dot{u}_n \rangle \, d\tau \leq \left| f \right|_{L^2(0,T;L^2(\Omega))} \left| u_n \right|_{L^2(0,T;L^2(\Omega))}
\]

\[
\leq \frac{1}{2} \left( \left| f \right|_{L^2(0,T;L^2(\Omega))}^2 + \left| \dot{u}_n \right|_{L^2(0,T;L^2(\Omega))}^2 \right).
\]

\[
\left(3.10\right)
\]

From \(3.7\), \(3.9\), and \(3.10\), we obtain that there exists \(c_7, c_8 > 0\) such that

\[
\left| \dot{u}_n(t) \right|_{L^2(\Omega)}^2 + c_2 \left| u_n(t) \right|_{H^1_0(\Omega)}^2 \leq c_7 + c_8 \int_0^t \left| u_n(\tau) \right|_{L^2(\Omega)}^2 \, d\tau \quad (t \in I_n),
\]

\[
\left(3.11\right)
\]

this implies that

\[
\left| \dot{u}_n(t) \right|_{L^2(\Omega)}^2 \leq c_7 + c_8 \int_0^t \left| u_n(\tau) \right|_{L^2(\Omega)}^2 \, d\tau \quad (t \in I_n).
\]

\[
\left(3.12\right)
\]

Using Gronwall’s inequality it follows that

\[
\left| \dot{u}_n(t) \right|_{L^2(\Omega)}^2 \leq c_7 \exp(c_8 t) \quad (t \in I_n).
\]

\[
\left(3.13\right)
\]
Therefore, from (3.9), (3.11), and (3.13), we get that there exists $c_9 > 0$,
\[
|\dot{u}_n(t)|_{L^2(\Omega)} \leq c_9, \quad |u_n(t)|_{L^2(\Omega)} \leq c_9, \quad |u_n(t)|_{H^1_0(\Omega)} \leq c_9, \quad (t \in T_n),
\]
(3.14)
where $c_4, c_5, c_6, c_7, c_8, c_9$ are positive constants independent of $n$ and $T_n$, from which we can assert that $T_n = [0, T] \ (\forall n)$.

For every $\eta \in \text{span}\{e_1, e_2, \ldots, e_n\}$, from (3.4)
\[
|\langle \ddot{u}_n, \eta \rangle| \leq |A(t)(\dot{u}_n)|_{L^2(\Omega)} \cdot |\eta|_{L^2(\Omega)} + |f(t)|_{L^2(\Omega)} \cdot |\eta|_{L^2(\Omega)} + |b_n(u_n)|_{L^2(\Omega)} \cdot |\eta|_{L^2(\Omega)} + |B| \cdot |u_n|_{H^1_0(\Omega)} \cdot |\eta|_{H^1_0(\Omega)},
\]
(3.15)
where $|B|$ is the norm of linear continuous operator $B$
\[
|\ddot{u}_n(t)|_{H^{-1}(\Omega)} = \sup_{|\eta|_1 = 1} |\langle \ddot{u}_n(t), \eta \rangle| = \sup_{|\eta|_1 = 1} |\langle \ddot{u}_n(t), \eta \rangle| \\ 
\leq c_{10} \left( |A(t)(\dot{u}_n)|_{L^2(\Omega)} + |f(t)|_{L^2(\Omega)} + |b_n(u_n)|_{L^2(\Omega)} + |B| \cdot |u_n(t)|_{H^1_0(\Omega)} \right),
\]
(3.16)
where $c_{10}$ is the imbedding constant which $H^1_0(\Omega)$ imbeds in $L^2(\Omega)$
\[
|b_n(u_n)(t)|_{L^2(\Omega)}^2 = \int_{\Omega} |b_n(u_n)(t)|^2 \, dx \leq \int_{\Omega} c^2(1 + |u_n(x,t)|)^2 \, dx \\ 
\leq 2c^2 \int_{\Omega} \left( 1 + |u_n(x,t)|^2 \right) dx = 2c^2 \left( |\Omega| + |u_n(t)|^2_{L^2(\Omega)} \right),
\]
(3.17)
this shows that $\{b_n(u_n)\}$ is also a bounded subset of $L^\infty(0,T;L^2(\Omega))$. Since $A$ is weakly continuous, it must be a bounded operator from $L^2(0,T;L^2(\Omega))$ to $L^2(0,T;L^2(\Omega))$. But $\{\ddot{u}_n\}$ is a bounded subset of $L^2(0,T;L^2(\Omega))$, $\{A(t)(\dot{u}_n)\}$ must be a bounded subset of $L^2(0,T;L^2(\Omega))$. Inequality (3.16) implies that $\{\ddot{u}_n\}$ is a bounded subset of $L^2(0,T;H^{-1}(\Omega))$.

Therefore, there exist a subsequence of $\{u_n\}$, still denoted by itself, and a function $u$ such that $u \in L^\infty(0,T;H^1_0(\Omega))$, $\dot{u} \in L^\infty(0,T;L^2(\Omega))$, $\ddot{u} \in L^2(0,T;H^{-1}(\Omega))$ satisfying
\[
u_n \rightarrow u \quad \text{weakly-star in } L^\infty(0,T;H^1_0(\Omega)),
\]
\[
\dot{u}_n \rightarrow \dot{u} \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)),
\]
\[
\ddot{u}_n \rightarrow \ddot{u} \quad \text{weakly in } L^2(0,T;L^{-1}(\Omega)),
\]
\[
b_n(u_n) \rightarrow g \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)).
\]
(3.18)
Since, the space $W(V)$ defined by $W(V) = \{ u \in L^2(0,T;V), \ \ddot{u} \in L^2(0,T;V') \}$ forms a real Hilbert space with the norm $|u|_W = |u|_{L^2(0,T;V')} + |\ddot{u}|_{L^2(0,T;V''')}$ and is continuously imbedded in $C([0,T];L^2(\Omega))$, it is obvious that $u \in C(0,T;L^2(\Omega))$, $\dot{u} \in C(0,T;H^{-1}(\Omega))$. Hence, $u(0)$, $\dot{u}(0)$ make sense.

For $\lambda \in L^2(0,T)$, from (3.4) we have
\[
\int_0^T \langle \ddot{u}_n, \lambda e_i \rangle \, dt = -\int_0^T \langle A(t)(\dot{u}_n), \lambda e_i \rangle \, dt - \int_0^T \langle B(u_n), \lambda e_i \rangle \, dt \\ 
- \int_0^T \langle b_n(u_n), \lambda e_i \rangle \, dt + \int_0^T \langle f(t), \lambda e_i \rangle \, dt, \quad i = 1, 2, \ldots, n.
\]
(3.19)
For every given positive integer \( i \), let \( n \to \infty \) in (3.19), it follows that

\[
\int_0^T \langle \ddot{u}, \lambda e_i \rangle \, dt = -\int_0^T \langle A(t) (\dot{u}), \lambda e_i \rangle \, dt - \int_0^T \langle B(u), \lambda e_i \rangle \, dt - \int_0^T \langle g, \lambda e_i \rangle \, dt + \int_0^T \langle f(t), \lambda e_i \rangle \, dt, \quad i = 1, 2, \ldots, n.
\] (3.20)

Therefore, we have from (3.20)

\[
\ddot{u}(t) + A(t) (\dot{u}) + B(u) + g(t) = f(t), \quad \text{in } L^2(0, T; H^{-1}(\Omega)).
\] (3.21)

Next, we demonstrate that

\[
g(x, t) \in \varphi(u(x, t)) \quad \text{a.e. } (x, t) \in Q_T = \Omega \times [0, T].
\] (3.22)

Since, \( u_n(x, t) \to u(x, t) \) a.e. \((x, t) \in Q_T\), by Eropl's theorem [9], for every \( \delta > 0 \), there exists a subset \( Q_\delta \subset Q_T = \Omega \times [0, T], |Q_\delta| \leq \delta \),

\[
u_n(x, t) \to u(x, t) \quad \text{uniformly in } Q_T \setminus Q_\delta
\] (3.23)

that is, for every \( \varepsilon > 0 \), there exists a positive integer \( N \), when \( n \geq N \),

\[
|u_n(x, t) - u(x, t)| \leq \varepsilon \quad \forall (x, t) \in Q_T \setminus Q_\delta
\] (3.24)

It is obvious that, when \( 1/n \leq \varepsilon \) and \( n \geq N \), for almost everywhere \((x, t) \in Q_T \setminus Q_\delta\)

\[
b_n(u_n(x, t)) = b_n(u_n(x, t)) = \tilde{b}_n(u_n(x, t)) \leq \tilde{b}_{2\varepsilon}(u(x, t)).
\] (3.25)

For every \( \mu \in L^1(0, T; L^2(\Omega)) \), \( \mu \geq 0 \)

\[
\int_{Q_T \setminus Q_\delta} g(x, t) \mu(x, t) \, dx \, dt = \lim_{n \to \infty} \int_{Q_T \setminus Q_\delta} b_n(u_n(x, t)) \mu(x, t) \, dx \, dt
\leq \int_{Q_T \setminus Q_\delta} \tilde{b}_{2\varepsilon}(u(x, t)) \mu(x, t) \, dx \, dt,
\] (3.26)

\[
\int_{Q_T \setminus Q_\delta} g(x, t) \mu(x, t) \, dx \, dt \leq \limsup_{\varepsilon \to 0^+} \int_{Q_T \setminus Q_\delta} \tilde{b}_{2\varepsilon}(u(x, t)) \mu(x, t) \, dx \, dt
\leq \int_{Q_T \setminus Q_\delta} \tilde{b}(u(x, t)) \mu(x, t) \, dx \, dt.
\] (3.27)

Analogously, we can obtain

\[
\int_{Q_T \setminus Q_\delta} g(x, t) \mu(x, t) \, dx \, dt \geq \int_{Q_T \setminus Q_\delta} b(u(x, t)) \mu(x, t) \, dx \, dt.
\] (3.27)

Hence, (3.26) and (3.27) imply that

\[
g(x, t) \in \varphi(u(x, t)) \quad \text{a.e. } (x, t) \in Q_T \setminus Q_\delta.
\] (3.28)

Finally, let \( \delta \to 0^+ \), we get

\[
g(x, t) \in \varphi(u(x, t)) \quad \text{a.e. } (x, t) \in Q_T = \Omega \times [0, T].
\] (3.29)
Let $\lambda \in C^{1}[0,T]$, $\lambda(T) = 0$, integrating by parts the left-hand side of equations (3.19) and (3.20) gives

$$
-\langle u_{n}(0),\lambda(0)e_{i} \rangle - \int_{0}^{T} \langle u_{n},\lambda e_{i} \rangle \, dt = -\int_{0}^{T} (A(t)(u_{n}),\lambda e_{i}) \, dt - \int_{0}^{T} \langle B(u_{n}),\lambda e_{i} \rangle \, dt \\
- \int_{0}^{T} \langle b_{n}(u_{n}),\lambda e_{i} \rangle \, dt - \int_{0}^{T} \langle f(t),\lambda e_{i} \rangle \, dt,
$$

(3.30)

and

$$
-\langle \dot{u}(0),\lambda(0)e_{i} \rangle - \int_{0}^{T} \langle u,\lambda e_{i} \rangle \, dt = -\int_{0}^{T} (A(t)(u),\lambda e_{i}) \, dt - \int_{0}^{T} \langle B(u),\lambda e_{i} \rangle \, dt \\
- \int_{0}^{T} \langle g,\lambda e_{i} \rangle \, dt - \int_{0}^{T} \langle f(t),\lambda e_{i} \rangle \, dt,
$$

(3.31)

making comparison between (3.30) and (3.31) we get that

$$
\lim_{n \to \infty} \langle \dot{u}_{n}(0) - \dot{u}(0),e_{i} \rangle = 0, \quad i = 1,2,\ldots,n
$$

(3.32)

therefore, this implies that

$$
\dot{u}_{n}(0) \to \dot{u}(0) \text{ weakly in } H^{-1}(\Omega)
$$

(3.33)

uniqueness of limit implies that $\dot{u}(0) = u_{1}$ (in $H^{-1}(\Omega)$).

Let $\lambda \in C^{2}[0,T]$, $\lambda(T) = 0$, $\lambda(T) = 0$. Analogously, integrating by parts the left-hand side of equations (3.30) and (3.31), and making comparison with the obtained results again gives: $u(0) = u_{0}$ (in $L^{2}(\Omega)$).

THEOREM 3.2. Let $f \in L^{2}(0,T;L^{2}(\Omega))$, $u_{0} \in H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$, $u_{1} \in L^{2}(\Omega)$. Assume that $b$ satisfies

(1') $b(\xi)\xi \geq 0$ for almost everywhere $\xi \in \mathbb{R}$, and $\exists \zeta > 0$,

$$
|b(\xi)| \leq \zeta (1 + |\xi|^{p}), \quad \text{a.e. } \xi \in \mathbb{R}, \text{ if } n > 2, \ 0 < p \leq \frac{2n}{n-2}; \text{ if } n \leq 2, \ 0 \leq p < \infty.
$$

(3.34)

Then, under assumptions (2), (3), and (4), there exists a function $v$ defined in $\Omega \times [0,T]$ satisfying

$$
\dot{v} + A(t)(\dot{v}) + B(v) + g(t) = f(t) \quad \text{in } L^{1}(0,T;H^{-1}(\Omega) + L^{1}(\Omega)),
$$

(3.35)

$$
\dot{g}(x,t) \in \varphi(v(x,t)) \quad \text{a.e. } (x,t) \in Q_{T} = \Omega \times [0,T],
$$

and

$$
\dot{v}(0) = u_{0}, \quad \dot{v}(0) = u_{1}.
$$

Proof. It is also easy to see that $b_{\varepsilon}$ satisfies assumption (1)' with possible different constant $\zeta$. Analogously to Theorem 3.1, we still may get (3.5), where $\{e_{n}\}_{n=1}^{\infty}$ is a basis of $H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$ satisfying $(e_{i},e_{j}) = \delta_{ij}$. Set

$$
J_{n}(\xi) = \int_{0}^{\xi} b_{n}(t) \, dt,
$$

(3.36)

then $J_{n}(\xi) \geq 0$, $\forall \xi \in \mathbb{R}$, and

$$
\int_{0}^{T} \langle b_{n}(u_{n}),\dot{u}_{n} \rangle \, dt = \int_{\Omega} J_{n}(u_{n}(x,t)) \, dx - \int_{\Omega} J_{n}(u_{n}(x,0)) \, dx \geq \int_{\Omega} J_{n}(u_{n}(x,t)) \, dx
$$

$$
\int_{0}^{T} |b_{n}(\xi)| \, dt \leq \int_{|z| \leq 1} |b(\xi - \frac{z}{n})|^{p} \, dz \leq d_{1} + d_{2} |\xi|^{p},
$$

(3.37)
where $d_1$ and $d_2$ are positive constants independent of $n$. 

$$ |J_n(x_n)| = \left| \int_0^{X_n} b_n(t) \, dt \right| \leq (\text{sgn} x_n) \cdot \int_0^{X_n} |b_n(t)| \, dt \leq (\text{sgn} x_n) \cdot \left( d_1 + d_2 |t|^p \right) dt = d_1 |x_n|^p + d_2 |x_n|^{p+1},$$

(3.38)

$$ \left| \int_\Omega J_n(x_n) \, dt \right| \leq \int_\Omega |J_n(x_n)| \, dx \leq d_1 |x_n|_{L^1(\Omega)} + \frac{d_2 |x_n|^{p+1}_{L^{p+1}(\Omega)}}{(p+1)},$$

Since $L^{p+1}(\Omega) \subset L^1(\Omega)$ and $u_n(0) = x_n \to u_0$ strongly in $L^{p+1}(\Omega)$, and $|x_n|_{L^1(\Omega)}$ are bounded, and so is $\int_\Omega J_n(x_n(x)) \, dx$. From (3.5) we have

$$ |\dot{u}_n(t)|^2_{L^2(\Omega)} + c_2 |u_n(t)|^2_{H^1_0(\Omega)} \leq c_4 + c_3 |u_n(t)|^2_{L^2(\Omega)} + 2 \int_0^t \langle f, \dot{u}_n \rangle \, dt.$$  

(3.39)

It is easy to see that (3.9), (3.10), (3.11), (3.13), and (3.14) are still true and the solution of (3.3) can be extended to interval $[0, T]$. By Sobolev imbedding theorem, we have, for a.e. $t \in [0, T]$, if $n > 2$, then $H^1_0(\Omega) \subset L^{p^*}(\Omega) \subset L^p(\Omega)$, $p^* = 2n/(n-2)$, and $|u_n(t)|_{L^p(\Omega)} \leq c_{10} |u_n(t)|_{H^1_0(\Omega)} \leq c_{10} c_9$; if $n = 2$, then $H^1_0(\Omega) \subset L^q(\Omega)$, when $1 \leq q < \infty$, so $|u_n(t)|_{L^q(\Omega)} \leq c_{10} |u_n(t)|_{H^1_0(\Omega)} \leq c_{10} c_9$; if $n = 1$, then $H^1_0(\Omega) \subset C(\bar{\Omega})$ and ditto, $|u_n(t)|_{C(\bar{\Omega})} = \max_{x \in \Omega} |u_n(x, t)| \leq c_{10} c_9$, where $\bar{\Omega}$ denotes the closure of $\Omega$ and $c_{10}$ is the imbedding constant which $H^1_0(\Omega)$ imbeds in $L^p(\Omega)$ or $C(\bar{\Omega})$. Note that, we always have that $b_n(u_n) \in L^\infty(0, T; L^{p_0}(\Omega))$, where $p_0 = (n+1)/(n-2)$ and $\{b_n(u_n)\}$ is a bounded subset of $L^\infty(0, T; L^{p_0}(\Omega))$. Therefore, there exist a subsequence of $\{u_n\}$, still denoted by itself, and a function $\nu$ such that $\nu \in L^\infty(0, T; H^1_0(\Omega))$, $\nu \in L^\infty(0, T; L^2(\Omega))$ satisfying

$$ u_n \rightharpoonup \nu \text{ weakly-star in } L^\infty(0, T; H^1_0(\Omega)), $$

$$ \dot{u}_n \rightharpoonup \dot{\nu} \text{ weakly-star in } L^\infty(0, T; L^2(\Omega)), $$

$$ b_n(u_n) \rightharpoonup g \text{ weakly-star in } L^\infty(0, T; L^{p_0}(\Omega)). $$

(3.40)

Since, the dual of the space $H^1_0(\Omega) \cap L^\infty(\Omega)$ is the space $L^1(0, T; H^{-1}(\Omega) + L^1(\Omega))$, it is easy to obtain from (3.4) that

$$ \dot{v}(t) + A(t) \dot{v} + B(v) + \dot{g}(t) = f(t) \text{ in } L^1(0, T; H^{-1}(\Omega) + L^1(\Omega)). $$

(3.41)

Analogous to Theorem 3.1, we can complete the proof of this theorem.

**Remark 3.3.** If $A(t) = A$ and $A$ is linear, then the uniqueness of such solution will be obtained in the same way as in [3].

### 4. Decay of the solution

**Theorem 4.1.** Let $T = +\infty$, $f \equiv 0$. Suppose that for every $t \geq 0$, the operator $A(t)$ satisfies

$$ (A(t)w, w) \geq \delta_0 |w|^2_{L^2(\Omega)}, \forall w \in L^2(\Omega). $$

(4.1)

Moreover, if $\langle Bw, w \rangle \geq 0, \forall w \in H^1_0(\Omega)$ or $c_3 c_{10} \leq c_2$, here $c_{10}$ is an imbedding constant which $H^1_0(\Omega)$ imbeds in $L^2(\Omega)$. Then, under conditions of Theorem 3.2, the solution in
Theorem 3.2 obtained from the regularized equation (3.3) satisfies

\[ |\dot{u}(t)|_{L^2(\Omega)}^2 \leq \mu_1 \exp(-\mu_2 t), \quad \text{a.e. } t \geq 0, \tag{4.2} \]

where \( \delta_0, \mu_1, \mu_2 \) are positive constants.

**Proof.** Let \( u_n \) be a solution of (3.3), that is, satisfies (3.4) and (3.5). Since \( J_n(u_n(x,t)) \geq 0 \) by (3.5) we have

\[ |\dot{u}(t)|_{L^2(\Omega)}^2 + \langle Bu_n(t), u_n(t) \rangle \leq c_{11} - 2\delta_0 \int_0^t |\dot{u}_n(\tau)|_{L^2(\Omega)}^2 d\tau, \quad t \in [0, +\infty), \tag{4.3} \]

where \( c_{11} \) is a positive constant independent of \( n \). If \( \langle Bw, w \rangle \geq 0 \), for every \( w \in H_0^1(\Omega) \), \( \langle Bu_n(t), u_n(t) \rangle \geq 0 \). Analogously to [7, Theorem 4], we obtain:

\[ |\dot{u}_n(t)|_{L^2(\Omega)}^2 \leq c_{11} \exp(-2\delta_0 t), \quad \text{a.e. } t \geq 0. \tag{4.4} \]

If \( c_3c_{10}^2 \leq c_2 \), we get from (4.3) that

\[ |\dot{u}_n(t)|_{L^2(\Omega)}^2 + c_2 |u_n(t)|_{H_0^1(\Omega)}^2 \leq c_{11} + c_3 |u_n(t)|_{L^2(\Omega)}^2 - 2\delta_0 \int_0^t |\dot{u}_n(\tau)|_{L^2(\Omega)}^2 d\tau \]

\[ \leq c_{11} + c_3 c_{10}^2 |u_n(t)|_{H_0^1(\Omega)}^2 - 2\delta_0 \int_0^t |\dot{u}_n(\tau)|_{L^2(\Omega)}^2 d\tau, \tag{4.5} \]

from which it is permitted to get inequality (4.4).

Since \( |\dot{u}_n(t)|_{L^2(\Omega)} \leq c_9, \dot{u} \rightarrow \dot{u} \) weakly-star in \( L^\infty(0, \infty; L^2(\Omega)) \), it is easy to obtain that \( \dot{u}(t) \rightarrow \dot{u}(t) \) weak in \( L^2(\Omega) \) for a.e. \( t \geq 0 \). But \( L^2(\Omega) \) is a real Hilbert space, therefore, \( |\dot{u}(t)|_{L^2(\Omega)} \leq \lim_{n \to \infty} |\dot{u}_n(t)|_{L^2(\Omega)}, \text{ a.e. } t \geq 0 \). Finally, we get \( |\dot{u}(t)|_{L^2(\Omega)} \leq c_{11} \exp(-2\delta_0 t) \), (a.e. \( t \geq 0 \)).

**References**


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