SOME SUFFICIENT CONDITIONS FOR STRONGLY STARLIKENESS

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Abstract. We consider strongly starlikeness of order \( \alpha \) of functions \( f(z) = z + a_{n+1}z^{n+1} + \cdots \) which are analytic in the unit disc and satisfy the condition of the form

\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \lambda, \quad 0 < \mu < 1, \; 0 < \lambda < 1.
\]

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1. Introduction and preliminaries. Let \( H \) denote the class of functions analytic in the unit disc \( U = \{ z : |z| < 1 \} \) and let \( A \subset H \) be the class of normalized analytic functions \( f \) in \( U \) such that \( f(0) = f'(0) - 1 = 0 \). For \( n \geq 1 \) we put

\[
A_n = \{ f : f(z) = z + a_{n+1}z^{n+1} + \cdots \text{ is analytic in } U \} \quad (1.1)
\]

and \( A_1 = A \).

A function \( f \in A \) is said to be strongly starlike of order \( \alpha \), \( 0 < \alpha \leq 1 \), if and only if

\[
z f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} < \left( \frac{1+\lambda z}{1-z} \right)^{\alpha}, \quad (1.2)
\]

where \( < \) denotes the usual subordination. We denote this class by \( S(\alpha) \). If \( \alpha = 1 \), then \( S(1) \equiv S^* \) is the well-known class of starlike functions in \( U \) (cf. [1]).

In this paper, we find a condition so that \( f \in A_n \) satisfying

\[
f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z, \quad 0 < \mu < 1, \; 0 < \lambda < 1, \quad (1.3)
\]

is in \( S(\alpha) \). Also, we consider an integral transformation.

We note that the author in [4] determined the values for \( \lambda \) in (1.3) which implies starlikeness in \( U \). Recently, Ponnusamy and Singh [5] found the condition which implies the strongly starlikeness of order \( \alpha \), but for \( \mu < 0 \) in (1.3). By using the similar method as in [5] we consider strongly starlikeness in the case (1.3).

First, we cite the following lemma.

**Lemma 1.1.** Let \( Q \in H \) satisfy the subordination condition

\[
Q(z) < 1 + \lambda_1 z, \quad Q(0) = 1, \quad (1.4)
\]

where \( 0 < \lambda_1 \leq 1 \). For \( 0 < \alpha \leq 1 \), let \( p \in H \), \( p(0) = 1 \) and \( p \) satisfy the condition

\[
Q(z)p^{\alpha}(z) < 1 + \lambda z, \quad 0 < \lambda \leq 1. \quad (1.5)
\]
Then for
\[
\sin^{-1} \lambda + \sin^{-1} \lambda_1 \leq \frac{\alpha \pi}{2}
\]  
(1.6)
we have \(\text{Re}\{p(z)\} > 0\) in \(U\).

This is the special case of the more general lemma given in [5].

2. Results and consequences. For our results we also need the following two lemmas.

**Lemma 2.1.** Let \(p \in H, p(z) = 1 + p_n z^n + \cdots, n \geq 1\), satisfy the condition
\[
p(z) - \frac{1}{\mu} z p'(z) < 1 + \lambda z, \quad 0 < \mu < 1, \quad 0 < \lambda \leq 1.
\]  
(2.1)
Then
\[
p(z) < 1 + \lambda_1 z,
\]  
(2.2)
where
\[
\lambda_1 = \frac{\lambda \mu}{n - \mu}.
\]  
(2.3)
The proof of this lemma for \(n = 1\) is given by [4]. For any \(n \in \mathbb{N}\) we also can apply Jack’s lemma [3].

**Lemma 2.2.** If \(0 < \mu < 1, 0 < \lambda \leq 1\) and \(Q \in H\) satisfying
\[
Q(z) < 1 + \frac{\lambda \mu}{n - \mu} z, \quad Q(0) = 1, \quad n \in \mathbb{N},
\]  
(2.4)
and if \(p \in H, p(0) = 1\) and satisfies
\[
Q(z) p^n(z) < 1 + \lambda z,
\]  
(2.5)
where
\[
0 < \lambda \leq \frac{(n - \mu) \sin(\pi \alpha/2)}{|\mu + (n - \mu) e^{i\pi \alpha/2}|},
\]  
(2.6)
then \(\text{Re}\{p(z)\} > 0\) in \(U\).

**Proof.** If in Lemma 1.1 we put \(\lambda_1 = \lambda \mu / (n - \mu)\), then the condition (1.6) is equivalent to
\[
\sin^{-1} \lambda + \sin^{-1} \frac{\lambda \mu}{n - \mu} \leq \frac{\alpha \pi}{2}.
\]  
(2.7)
This inequality is equivalent to
\[
\sin^{-1} \left( \lambda \sqrt{1 - \frac{\lambda^2 \mu^2}{(n - \mu)^2}} + \frac{\lambda \mu}{n - \mu} \sqrt{1 - \lambda^2} \right) \leq \frac{\alpha \pi}{2},
\]  
(2.8)
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or to the inequality
\[ \lambda \left[ \sqrt{(n - \mu)^2 - \lambda^2} + \mu \sqrt{1 - \lambda^2} \right] \leq (n - \mu) \sin \left( \frac{\alpha \pi}{2} \right). \] (2.9)

From there, after some transformations, we get the following equivalent inequality
\[ \left\{ \left[ \mu^2 + (n - \mu)^2 \right]^2 - 4\mu^2(n - \mu)^2 \cos^2 \left( \frac{\alpha \pi}{2} \right) \right\} \lambda^4 - 2(n - \mu)^2 \left[ \mu^2 + (n - \mu)^2 \right] \sin^2 \left( \frac{\alpha \pi}{2} \right) \lambda^2 + (1 - \mu)^4 \sin^4 \left( \frac{\alpha \pi}{2} \right) \geq 0 \] (2.10)

which is equivalent to the condition (2.6).

By Lemma 1.1 we have that \( \Re \{ p(z) \} > 0 \) in \( U \).

**Theorem 2.3.** Let \( f \in A_n, 0 < \mu < 1 \) and \( f \) satisfy the subordination
\[ f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z, \] (2.11)

where
\[ 0 < \lambda \leq \frac{n - \mu}{\sqrt{\mu^2 + (n - \mu)^2}}. \] (2.12)

Then \( f \in S^* \).

**Proof.** If we put \( Q(z) = (z/f(z))^\mu = 1 + q_n z^n + \cdots \), then after some calculations, we get
\[ Q(z) - \frac{1}{\mu} z Q'(z) = f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z. \] (2.13)

From Lemma 2.1 we have
\[ Q(z) < 1 + \lambda_1 z, \quad \lambda_1 = \frac{\lambda \mu}{n - \mu}. \] (2.14)

The rest part of the proof is the same as in the case \( n = 1 \) (see [4, Theorem 1]) and we omit the details.

**Theorem 2.4.** Let \( 0 < \mu < 1 \) and \( 0 < \alpha \leq 1 \). If \( f \in A_n \) satisfies
\[ \left| f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \frac{(n - \mu) \sin(\pi \alpha/2)}{\mu + (n - \mu) e^{i\pi \alpha/2}}, \quad z \in U, \] (2.15)

then \( f \in S(\alpha) \).

**Proof.** If we put \( \lambda = \frac{(n - \mu) \sin(\pi \alpha/2)}{\mu + (n - \mu) e^{i\pi \alpha/2}} \), then, since \( 0 < \alpha \leq 1 \), we have \( 0 < \lambda \leq (n - \mu)/\sqrt{\mu^2 + (n - \mu)^2} \), and by Theorem 2.3, \( f \in S^* \). Later, the function \( Q(z) = (z/f(z))^\mu = 1 + q_n z^n + \cdots \) is analytic in \( U \) and satisfies \( Q(z) < 1 + \lambda_1 z \), \( \lambda_1 = \lambda \mu/(n - \mu) \). Now, if we define
\[ p(z) = \left( \frac{zf'(z)}{f(z)} \right)^{1/\alpha}, \] (2.16)
then \( p \) is analytic in \( U \), \( p(0) = 1 \) and condition (2.15) is equivalent to
\[
Q(z)p^\alpha(z) < 1 + \lambda z. \tag{2.17}
\]

Finally, from Lemma 2.2 we obtain
\[
\left( \frac{zf'(z)}{f(z)} \right)^{1/\alpha} < \frac{1 + z}{1 - z} \quad \left( \iff \frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^\alpha \right), \tag{2.18}
\]
that is, \( f \in S(\alpha) \).

We note that for \( \alpha = 1 \) we have the statement of Theorem 2.3.
For \( n = 1, \mu = 1/2, \alpha = 2/3 \) we get the following corollary.

**COROLLARY 2.5.** Let \( f \in A \) and let
\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^{3/2} - 1 \right| < \frac{1}{2}, \quad z \in U. \tag{2.19}
\]
Then
\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{3}, \quad z \in U, \tag{2.20}
\]
that is, \( f \in S(2/3) \).

**THEOREM 2.6.** Let \( 0 < \mu < 1, \Re \{ c \} > -\mu, \) and \( 0 < \alpha \leq 1 \). If \( f \in A_n \) satisfies
\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \left| \frac{n+c-\mu}{c-\mu} \right| \left| \frac{n-\mu}{\mu+(n-\mu)e^{i\pi\alpha/2}} \right|, \quad z \in U, \tag{2.21}
\]
then the function
\[
F(z) = z \left[ \frac{c-\mu}{z^{c-\mu}} \int_0^z \left( \frac{t}{f(t)} \right)^\mu t^{c-\mu-1} dt \right]^{-1/\mu} \tag{2.22}
\]
belongs to \( S(\alpha) \).

**PROOF.** If we put that \( \lambda \) is equal to the right-hand side of the inequality (2.21) and
\[
Q(z) = F'(z) \left( \frac{z}{F(z)} \right)^{1+\mu} = 1 + q_n z^n + \cdots, \tag{2.23}
\]
then from (2.21) and (2.22) we obtain
\[
Q(z) + \frac{1}{c-\mu} z Q'(z) = f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z. \tag{2.24}
\]
Hence, by using the result of Hallenbeck and Ruscheweyh [2, Theorem 1] we have that
\[
Q(z) < 1 + \lambda_1 z, \quad \lambda_1 = \left| \frac{(c-\mu)\lambda}{n+c-\mu} \right| \left| \frac{(n-\mu)\sin(\pi\alpha/2)}{\mu+(n-\mu)e^{i\pi\alpha/2}} \right|, \tag{2.25}
\]
and the desired result easily follows from Theorem 2.4. \( \square \)
**Remark 2.7.** For $\alpha = 1$ and $n = 1$ we have the corresponding result given earlier in [4]. For $c = \mu + 1$, we have

**Corollary 2.8.** Let $0 < \mu < 1$ and $0 < \alpha \leq 1$. If $f \in A_n$ satisfies the condition

$$
\left| f'(z) \left( \frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \frac{n(n-\mu) \sin(\pi \alpha/2)}{|\mu + (n-\mu)e^{i\pi \alpha/2}|}, \quad z \in U,
$$

(2.26)

then the function

$$
F(z) = z \left[ \frac{1}{z} \int_0^z \left( \frac{t}{f(t)} \right) \mu \, dt \right]^{-1/\mu}
$$

(2.27)

belongs to $S(\alpha)$.

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**References**


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