SUBORDINATION BY CONVEX FUNCTIONS

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(Received 18 January 1990)

Abstract. Let $K(\alpha)$, $0 \leq \alpha < 1$, denote the class of functions $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular and univalently convex of order $\alpha$ in the unit disc $U$. Pursuing the problem initiated by Robinson in the present paper, among other things, we prove that if $f$ is regular in $U$, $f(0) = 0$, and $f(z) + zf'(z) < g(z) + zg'(z)$ in $U$, then (i) $f(z) < g(z)$ at least in $|z| < r_0$, $r_0 = \sqrt{5/3} = 0.745...$ if $f \in K$; and (ii) $f(z) < g(z)$ at least in $|z| < r_1$, $r_1 = (51 - 24\sqrt{2})/23)^{1/2} = 0.8612...$ if $g \in K(1/2)$.

Keywords and phrases. Subordination, convex function, convex function of order $1/2$.

2000 Mathematics Subject Classification. Primary 30C45.

1. Introduction. Let $S$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are regular and univalent in the unit disc $U = \{z/|z| < 1\}$. For a given $\alpha$, $0 \leq \alpha < 1$, denote by $K(\alpha)$ the subclass of $S$ consisting of functions $f$ which satisfy the condition

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in U.$$  (1.1)

$K(\alpha)$ is called the class of convex functions of order $\alpha$ and $K = K(0)$ is the class of convex functions.

Suppose that $f$ and $g$ are regular in $|z| < \rho$ and $f(0) = g(0)$. In addition, suppose that $g$ is also univalent in $|z| < \rho$. We say that $f$ is subordinate to $g$ in $|z| < \rho$ (in symbols, $f(z) < g(z)$ in $|z| < \rho$) if $f(|z| < \rho) \subset g(|z| < \rho)$.

In 1947, Robinson [2] proved that if $g(z) + zg'(z)$ is in $S$ and $f(z) + zf'(z) < g(z) + zg'(z)$ in $|z| < 1$, then $f(z) < g(z)$ at least in $|z| < r_0 = 1/5$. Subsequently, Singh and Singh [4] increased the constant $r_0$ to $2 - \sqrt{3} = 0.268...$ Miller, Mocanu, and Read [1] further increased the constant to $4 - \sqrt{13} = 0.3944...$.

Here, we consider the problem of Robinson when $g \in K$ and $K(1/2)$, respectively. (It is easy to see that $g(z) + zg'(z)$ is close-to-convex and hence univalent in $|z| < 1$ when $g \in K$.) We remark that our method works even when $g \in K(\alpha)$. However, calculations in this general case become so cumbersome that the result obtained does not commend with the input labour. We, therefore, confine ourselves to the particular cases $\alpha = 0$ and $\alpha = 1/2$.

2. Preliminaries. We need the following results.

Lemma 2.1. Suppose that $f$ and $g$ are regular in $U$, $f(0) = g(0)$, and $g'(0) \neq 0$. Suppose further that

$$\text{Re} \left( 1 + \frac{zg''(z)}{g'(z)} \right) > -\frac{1}{2}, \quad z \in U.$$  (2.1)
Then if \( f(z) < g(z) \) in \( U \), we have
\[
\frac{1}{z} \int_0^z f(t) \, dt < \frac{1}{z} \int_0^z g(t) \, dt, \quad z \in U.
\tag{2.2}
\]

We observe that (2.1) implies that \( g \) is close-to-convex and hence univalent in \( U \) and that the right-hand side function in (2.2) is convex in \( U \) [3]. Lemma 2.1 is due to Miller, Mocanu, and Reade [1].

The underlying idea of the following result is essentially due to Zomorvić [6] (also, see [5]).

**Lemma 2.2.** Let \( P \) be regular in \( U \), \( P(0) = 1 \), and \( \text{Re} \, P(z) > 0 \) in \( U \). Let \( \mu \) and \( \lambda \) be fixed real numbers, \(-\infty < \mu < \infty, \lambda \geq 0, \text{ and } |z| = r < 1.\) Then
\[
\text{Re} \left[ \frac{\mu P(z) + \frac{zP'(z)}{P(z) + \lambda}}{P(z) + \lambda} \right] \geq \begin{cases} 
-\left( \sqrt{\lambda(\mu + 1)} - \sqrt{a + \lambda} \right)^2, & \text{if } \frac{\lambda(a + \lambda)}{(a - \rho + \lambda)^2} \geq \mu + 1 \\
(a - \rho) \left( \mu - \frac{\rho}{a - \rho + \lambda} \right), & \text{if } \mu + 1 > \frac{\lambda(a + \lambda)}{(a - \rho + \lambda)^2}, \\
(a + \rho) \left( \mu + \frac{\rho}{a + \rho + \lambda} \right), & \text{if } \mu + 1 < \frac{\lambda(a + \lambda)}{(a + \rho + \lambda)^2},
\end{cases}
\tag{2.3}
\]

where \( a = (1 + r^2)/(1 - r^2) \) and \( \rho = 2r/(1 - r^2). \)

**Proof.** Making use of the inequality (2.3) (see [5])
\[
\left| zP'(z) - \frac{P^2(z) - 1}{2} \right| \leq \frac{\rho^2 - \rho_0^2}{2},
\tag{2.4}
\]
where \( |P(z) - a| = \rho_0 \leq \rho \), we get
\[
\text{Re} \left[ \frac{\mu P(z) + \frac{zP'(z)}{P(z) + \lambda}}{P(z) + \lambda} \right] \geq \text{Re} \left[ \mu P(z) + \frac{P(z) - \lambda}{2} + \frac{(\lambda^2 - 1)(P(z) + \lambda)}{2|P(z) + \lambda|^2} \right] - \frac{\rho^2 - \rho_0^2}{2|P(z) + \lambda|^2}.
\tag{2.5}
\]

Taking \( P(z) = a + \xi + i\eta \) and \( R_1^2 = (a + \xi + \lambda)^2 + \eta^2 \), we get
\[
\text{Re} \left[ \frac{\mu P(z) + \frac{zP'(z)}{P(z) + \lambda}}{P(z) + \lambda} \right] \geq \mu(a + \xi) + \frac{a + \xi - \lambda}{2} + \frac{(\lambda^2 - 1)(a + \xi + \lambda)}{2R_1^2} - \frac{\rho^2 - \xi^2 - \eta^2}{2R_1}
= S(\xi, \eta).
\tag{2.6}
\]

Now it is easy to see that \( \partial S(\xi, \eta)/\partial \eta = 0 \) and \( \partial^2 S(\xi, \eta)/\partial \eta^2 > 0 \) at \( \eta = 0. \) Therefore,
\[
\min_{\eta} S(\xi, \eta) = S(\xi, 0)
= \mu(a + \xi) + \frac{a + \xi - \lambda}{2} + \frac{\lambda^2 - 1}{2(a + \xi + \lambda)} - \frac{\rho^2 - \xi^2}{2(a + \xi + \lambda)}
= (\mu + 1)R + \frac{\lambda(a + \lambda)}{R} - (\mu + 2)\lambda - a
= L(R),
\tag{2.7}
\]
where $R = a + \xi + \lambda$. Now, using the fact that $|R(z) - a| < \rho$, we obtain the inequality

$$a - \rho + \lambda \leq R \leq a + \rho + \lambda. \quad (2.8)$$

It is observed that at $R = R_0 = (\lambda(a + \lambda)/(\mu + 1))^{1/2}$, $\partial L(R)/\partial R = 0$ and $\partial^2 L(R)/\partial R^2 > 0$. Thus, $R = R_0$ gives the minimum value of $L(R)$ provided $R_0$ lies in the range of $R$. In view of (2.8), this is the case if the inequality

$$\frac{\lambda(a + \lambda)}{(a - \rho + \lambda)^2} \geq \mu + 1 \geq \frac{\lambda(a + \lambda)}{(a + \rho + \lambda)^2} \quad (2.9)$$

is satisfied. Thus, if (2.9) holds, we have

$$\min_R L(R) = L(R_0) = -\left(\sqrt{\lambda(\mu + 1)} - \sqrt{\lambda + a}\right)^2. \quad (2.10)$$

Also, it is easy to check that when $\mu + 1 > \lambda(a + \lambda)/(a - \rho + \lambda)^2$, $L(R)$ is an increasing function of $R$. Therefore, in this case,

$$\min_R L(R) = L(a - \rho + \lambda) = (a - \rho)\left(\mu - \frac{\rho}{a - \rho + \lambda}\right). \quad (2.11)$$

On the other hand, when $\mu + 1 < \lambda(a + \lambda)/(a + \rho + \lambda)^2$, $L(R)$ is a decreasing function of $R$. Therefore, in this case,

$$\min_R L(R) = L(a + \rho + \lambda) = (a + \rho)\left(\mu + \frac{\rho}{a + \rho + \lambda}\right). \quad (2.12)$$

This completes the proof of Lemma 2.2.

3. Theorems and their proofs

**Theorem 3.1.** Let $f$ be regular in $U$ with $f(0) = 0$ and let $g \in K$. Suppose that

$$f(z) + zf'(z) \prec g(z) + zg'(z) \quad \text{in } U. \quad (3.1)$$

Then $f(z) < g(z)$ at least in $|z| < r_0$, where $r_0 = \sqrt{5}/3 = 0.745\ldots$.

**Proof.** Let us take

$$h(z) = g(z) + zg'(z). \quad (3.2)$$

Since $g \in K$, we can put

$$1 + \frac{zh''(z)}{h'(z)} = P(z), \quad (3.3)$$

where $P(z)$ is regular in $U$, $P(0) = 1$, and $\Re P(z) > 0$ in $U$. Now, from (3.2) and (3.3), we get

$$1 + \frac{zh''(z)}{h'(z)} = P(z) + \frac{zP'(z)}{P(z) + 1}. \quad (3.4)$$
Taking $\mu = \lambda = 1$ in Lemma 2.2, we easily obtain
\[
\Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) \geq \begin{cases} 
\frac{1-2r}{1+r}, & \text{if } 0 \leq r < \frac{3}{5}, \\
-2\left[ 1 - \frac{a}{\sqrt{1-r^2}} \right]^2, & \text{if } \frac{3}{5} \leq r < 1,
\end{cases}
\] (3.5)

where $|z| = r < 1$. Now, it is easy to verify that for $0 \leq r < 3/5$, $\Re(1 + zh''(z)/h'(z)) > -1/2$ and for $3/5 \leq r < 1$, $\Re(1 + zh''(z)/h'(z)) > -1/2$ whenever $9r^4 + 22r^2 - 15 < 0$ or whenever $r < r_0$, where $r_0 = \sqrt{5}/3$ is the smallest positive root of $9r^4 + 22r^2 - 15 = 0$. The assertion of our theorem now follows from Lemma 2.1.

**Theorem 3.2.** Let $f$ be regular in $U$ with $f(0) = 0$ and let $g \in K(1/2)$. Suppose that
\[
f(z) + zf'(z) < g(z) + zg'(z) \quad \text{in } U.
\] (3.6)

Then
\[
f(z) < g(z)
\] (3.7)

at least in $|z| < r_1$, where $r_1 = ((51 - 24\sqrt{2})/23)^{1/2} = 0.8612\ldots$.

**Proof.** Let us put
\[
h(z) = g(z) + zg'(z).
\] (3.8)

Since $g \in K(1/2)$, we can write
\[
1 + \frac{zh''(z)}{h'(z)} = \frac{P(z) + 1}{2},
\] (3.9)

where $P(z)$ is regular in $U$, $P(0) = 1$, and $\Re P(z) > 0$ in $U$. From (3.8) and (3.9), we obtain
\[
1 + \frac{zh''(z)}{h'(z)} = \frac{1}{2} + \frac{P(z)}{2} + \frac{zP'(z)}{P(z) + 3}.
\] (3.10)

Using Lemma 2.2 (with $\mu = 1/2$ and $\lambda = 3$), we obtain, after some calculations,
\[
\Re \left[ 1 + \frac{zh''(z)}{h'(z)} \right] \geq \begin{cases} 
\frac{2}{(1+r)(2+r)}, & \text{if } 0 \leq r < \frac{-1 + \sqrt{5}}{2}, \\
6\left[ \frac{2-r^2}{1-r^2} \right]^{1/2} - 2\left( \frac{4-3r^2}{1-r^2} \right), & \text{if } \frac{-1 + \sqrt{5}}{2} \leq r < 1,
\end{cases}
\] (3.11)

where $|z| = r < 1$.

Now, we can easily check that for $0 \leq r < (-1 + \sqrt{5})/2$, $\Re(1 + zh''(z)/h'(z)) > -1/2$ and for $(-1 + \sqrt{5})/2 \leq r < 1$, $\Re(1 + zh''(z)/h'(z)) > -1/2$ whenever $23r^4 - 102r^2 + 63 > 0$ or whenever $r < r_1$, where $r_1 = ((51 - 24\sqrt{2})/23)^{1/2}$ is the smallest positive root of $23r^4 - 102r^2 + 63 = 0$. The desired result now follows from Lemma 2.1.

In the following theorem, we take for $g$ some distinguished members of $K$.

**Theorem 3.3.** Let $f$ be regular in $U$ with $f(0) = 0$ and let $f(z) + zf'(z) < g(z) + zg'(z)$ in $U$. Then
(a) $f(z) < g(z)$ in $U$ if $g(z) = z/(1-z)$;
(b) \( f(z) < g(z) \) at least in \( |z| < \rho_1 = ((28 - 8\sqrt{7})/7)^{1/2} = 0.98 \ldots \) if \( g(z) = -\log(1-z) \);
(c) \( f(z) < g(z) \) in \( U \) if \( g(z) = z + \lambda z^2 \), \( |\lambda| \leq 1/5 \);
(d) \( f(z) < g(z) \) at least in \( |z| < \rho_2 = (9 - \sqrt{33})/4 = 0.8138 \ldots \) if \( g(z) = e^z - 1 \).

We observe that the functions \( g \) defined in (a), (b), (c), and (d) belong to \( K, K(1/2), K(1/3), \) and \( K \), respectively.

**Proof.** We omit the proofs of parts (a), (c), and (d) and proceed to prove part (b).

Let \( h(z) = g(z) + zg'(z) \), where \( g(z) = -\log(1-z) \). Then \( h(0) = 0 \) and \( h'(0) \neq 0 \).

A simple computation shows that the condition
\[
\Re \left( 1 + \frac{2h''(z)}{h'(z)} \right) > \frac{-1}{2}
\]
(3.12)
is equivalent to
\[
\Re \left[ \frac{2}{(1-z)(2-z)} + \frac{1}{2} \right] > 0.
\]
(3.13)

If we let \( z = re^{i\theta} \), \( 0 \leq r < 1 \) and \( 0 \leq \theta \leq 2\pi \), then condition (3.13) takes the form
\[
\varphi(x) = 16r^2 x^2 - 6r(4 + r^2)x + r^4 + r^2 + 12 > 0,
\]
(3.14)

where \( x = \cos \theta \), \( 0 \leq \theta \leq 2\pi \). For \( r = 0 \), (3.14) is obviously satisfied. We, therefore, let \( r \neq 0 \).

Now, it can be readily verified that at \( x = x_0 = (12 + 3r^2)/16r \), we have \( \varphi'(x) = 0 \) and \( \varphi''(x) > 0 \).

Thus, \( x = x_0 \) gives the minimum value of \( \varphi(x) \) provided \(-1 \leq x_0 \leq 1 \). This is true if \( r \geq r_0 = (8 - \sqrt{28})/3 = 0.9028 \ldots \). Therefore, for \( r \in [r_0, 1] \),
\[
\min_{x \in [-1,1]} \varphi(x) = \varphi(x_0) = \frac{7r^4 - 56r^2 + 48}{16}.
\]
(3.15)

Hence, in this case, (3.14) is satisfied if \( 7r^4 - 56r^2 + 48 > 0 \), i.e., if \( r < \rho_1 = ((28 - 8\sqrt{7})/7)^{1/2} = 0.98 \ldots \). Also, for \( r \in [0, r_0] \), we can easily verify that \( \varphi(x) \) is a decreasing function of \( x \). Hence, in this case,
\[
\min_{x \in [-1,1]} \varphi(x) = \varphi(1) = 4 - 6^3 + 17^2 - 24 + 12
\]
\[
= (1-r)(2-r)(r^2-3r+6) > 0.
\]
(3.16)

Therefore, we conclude that for \( 0 \leq r < \rho_1 \),
\[
\Re \left( 1 + \frac{2h''(z)}{h'(z)} \right) > -\frac{1}{2}.
\]
(3.17)

Conclusion (b) now follows in view of Lemma 2.1.

**References**


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