A REMARK ON GWINNER’S EXISTENCE THEOREM ON VARIATIONAL INEQUALITY PROBLEM

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Abstract. Gwinner (1981) proved an existence theorem for a variational inequality problem involving an upper semicontinuous multifunction with compact convex values. The aim of this paper is to solve this problem for a multifunction with open inverse values.

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1. Introduction. In 1981, Gwinner [1] proved an existence theorem for a variational inequality problem, which is an infinite dimensional version of Walras excess demand theorem (see also Zedzler [5]).

Theorem 1.1. Let $P$ and $Q$ be nonempty compact convex subsets of locally convex Hausdorff topological vector spaces $X$ and $Y$, respectively. Let $f : P \times Q \to \mathbb{R}$ be continuous. Let $S : P \to Q$ be a multifunction. Suppose that

(i) for each $y \in Q, \{ x \in P : f(x, y) < t \}$ is convex for all $t \in \mathbb{R}$,
(ii) $S$ is an upper semicontinuous multifunction with nonempty compact convex values. Then there exist $x_0 \in P$ and $y_0 \in S(x_0)$ such that $f(x_0, y_0) \leq f(x, y_0)$ for all $x \in P$.

In this paper, our aim is to obtain the above variational inequality for a multifunction with open inverse values. We follow the method of Tarafdar and Yuan [4].

2. Preliminaries. In $N \in \mathbb{N}$, let $\langle N \rangle$ be the set of all nonempty subsets of $\{0, 1, 2, \ldots, N\}$, $\Delta_N = \text{co}\{e_0, e_1, \ldots, e_N\}$ be the standard simplex of dimension $N$, where $\{e_0, e_1, \ldots, e_N\}$ is the canonical basis of $\mathbb{R}^{N+1}$, and for $J \in \langle N \rangle$, let $\Delta_J = \text{co}\{e_j : j \in J\}$. Horváth [2] proved the following result.

Lemma 2.1. Let $X$ be a topological space and $F : \langle N \rangle \to X$. For each $J \in \langle N \rangle$, let $F(J)$ be a nonempty contractible subset of $X$ and for all $J, J' \in \langle N \rangle$ such that $J \subseteq J'$, suppose that $F(J) \subseteq F(J')$. Then there exists a continuous function $f : \Delta_N \to X$ such that $f(\Delta_J) \subseteq F(J)$ for all $J \in \langle N \rangle$.

Also, we need the following fixed point theorem due to Lassonde [4].

Lemma 2.2. Let $F : \Delta_N \to \Delta_N$ be a multifunction such that $F = F_n \circ F_{n-1} \circ \cdots \circ F_1 \circ F_0$, $\Delta_N \xrightarrow{F_0} X_1 \xrightarrow{F_1} X_2 \xrightarrow{F_2} \cdots \xrightarrow{F_n} X_{n+1} = \Delta_N$, where each $F_i$ is either a single-valued continuous function (in which case $X_{i+1}$ is assumed to be a Hausdorff topological space)
or an upper semicontinuous multifunction with $F_i(x)$, a nonempty compact convex subset of $X_{i+1}$ (in which case $X_{i+1}$ is a convex subset of a Hausdorff topological vector space). Then $F$ has a fixed point.

3. Main theorem

**Theorem 3.1.** Let $P$ as in Theorem 1.1 and $Q$ be an arbitrary subset of a locally convex Hausdorff topological vector space $Y$. Let $f : P \times Q \rightarrow \mathbb{R}$ be continuous and satisfy condition (i) of Theorem 1.1. Let $S : P \rightarrow Q$ be a multifunction such that

(i) $S^{-1}(X)$ is open for all $x \in Q$;

(ii) for each open set $F \subset P$, the set $\bigcap_{y \in F} S(y)$ is empty or contractible;

(iii) $S(P)$ is compact and contractible. Then the conclusion of Theorem 1.1 holds.

**Proof.** Since $P$ is compact, there exists a finite subset $\{x_0, x_1, x_2, \ldots, x_N\}$ of $S(P)$ such that $P = \bigcup_{i=0}^N S^{-1}(x_i)$. Define $F : \langle N \rangle \rightarrow S(P)$ by

$$F(J) = \begin{cases} \bigcap \{S(y) : y \in \bigcap_{j \in J} S^{-1}(x_j)\} & \text{if } \bigcap_{j \in J} S^{-1}(x_j) \neq \emptyset, \\ S(P) & \text{otherwise.} \end{cases}$$

(3.3.1)

It is clear that if $y \in \bigcap_{j \in J} S^{-1}(x_j)$, then $x_j \in S(y)$ for all $j \in J$. Thus, $F(J)$ is nonempty and contractible. Further, $F(J) \subseteq F(J')$ whenever $J \subseteq J'$. By Lemma 2.1, there exists a continuous function $f : \Delta_N \rightarrow \mathbb{R}$ such that $f(\Delta_j) \subseteq F(J)$ for all $J \in \langle N \rangle$. Let $\{g_i : i \in \{0, 1, 2, \ldots, N\}\}$ be a continuous partition of unity subordinated to the covering $\{S^{-1}(x_i) : i \in \{0, 1, \ldots, N\}\}$, that is, for each $i, g_i : P \rightarrow [0, 1]$ is continuous, $\{y \in P : g_i(y) \neq 0\} \subset S^{-1}(x_i)$, and $\sum_{i=0}^N g_i(y) = 1$ for all $y \in P$. Now, define $g : P \rightarrow \Delta_N$ by $g(y) = (g_0(y), g_1(y), \ldots, g_N(y))$ for all $y \in P$. Then $g$ is continuous. Further, $g(y) \in \Delta_f(y)$ for all $y \in P$, where $f(y) = \{i : g_i(y) \neq 0\}$. Therefore, $f \circ g(y) \in f(\Delta_f(y)) \subset F_f(y) \subset S(y)$.

Consider $T : S(P) \rightarrow P$ defined by $T(y) = \{z \in P : f(z, y) \leq f(w, y) \text{ for all } w \in P\}$. For each $y \in S(P)$, $T(y)$ is nonempty since $f$ assumes its minimum on the compact set $P$. Also, it is closed and hence compact. Further, $T(y)$ is convex. Indeed, let $z_1$ and $z_2 \in P$ be such that $f(z_i, y) \leq f(w, y)$ for all $w \in P$ and $i = 1, 2$. By the assumption on $f$, $f(\lambda z_1 + (1-\lambda) z_2, y) \leq f(w, y)$ for all $w \in P$. Since $f$ is continuous, the graph of $T, \text{Gr}(T) = \{(y, z) : y \in S(P), z \in T(y)\}$ is a closed subset of the compact set $S(P) \times P$. Then it follows that $T$ is upper semicontinuous.

Consider $G : \Delta_N \rightarrow \Delta_N$. Now, by Lemma 2.2, there exists $z_0 \in \Delta_N$ such that $z_0 \in G(z_0)$. Let $y_0 = f(z_0)$. Then $y_0 \in f \circ g \circ T \circ f(z_0)$, that is, there exists $x_0 \in T(y_0)$ so that $y_0 \in f \circ g(x_0) \in S(x_0)$. This completes the proof. □

**References**


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