APPROXIMATING FIXED POINTS OF $\lambda$-FIRMLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. We study the convergence of the Ishikawa iteration methods to fixed points for the result of Smarzewski (1991). Our theorems also improve recent theorems due to Sharma and Sahu (1996).

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1. Introduction. Let $E$ be a real Banach space and let $C$ be a nonempty closed convex subset of $E$. Then a mapping $T$ of $C$ into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T$ of $C$ into itself is called $\lambda$-firmly nonexpansive if there exists $\lambda \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \|(1 - \lambda)(x - y) + \lambda(Tx - Ty)\| \quad \forall x, y \in C. \quad (1.1)$$

It is clear that every $\lambda$-firmly nonexpansive mapping is nonexpansive. For a mapping $T$ of $C$ into itself, we consider the following iteration scheme: $x_1 \in C$,

$$x_{n+1} = \alpha_n T[y_n + (1 - \alpha_n)x_n] + (1 - \alpha_n)x_n \quad \forall n \geq 1, \quad (1.2)$$

where $\{\alpha_n\}$ are real sequences in $(0, 1]$ such that $0 < \alpha_n < 1$. Such an iteration scheme was introduced by Ishikawa [5]; see also Mann iteration scheme (corresponding to the choice $\beta_n = 0$ for all $n \in \mathbb{N}$) [6]. Now let $C$ be a nonempty convex subset of a Banach space $E$, and let $T, S$ be mappings of $C$ into itself. Then, for an $x_1 \in C$, we consider the iterates $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n T[y_n + (1 - \alpha_n)x_n],$$
$$y_n = \beta_n T[x_n + (1 - \beta_n)x_n] \quad \forall n \geq 1, \quad (1.3)$$

where $\alpha_n$ and $\beta_n$ are real sequences in $(0, 1]$. If $S = I$, the identity mapping, the iterates (1.3) are reduced to the special case of Ishikawa [5]. In 1991, Smarzewski [10] proved the following result: let $E$ be a uniformly convex Banach space and let $C = \bigcup_{i=1}^{n} C_i$ be a union of nonempty bounded closed convex subsets $C_i$ of $E$ and suppose $T : C \to C$ is $\lambda$-firmly nonexpansive for some $\lambda \in (0, 1)$. Then $T$ has a fixed point in $C$. The result above is no longer true if $T$ is merely nonexpansive, even in one-dimensional space; see [10]. Recently, Sharma and Sahu [9] studied the
convergence of the Mann and Ishikawa iteration methods to fixed points for the result of Smarzewski [10].

In this paper, we first show that the iterates \( \{x_n\} \) and \( \{y_n\} \) defined by (1.3) converge weakly to the same common fixed point of \( T \) and \( S \) when \( E \) is a uniformly convex Banach space with Opial’s condition or Fréchet differentiable norm. Next, we show that the iterates \( \{x_n\} \) defined by (1.2) converge weakly to a fixed point of \( T \) when \( E \) is a uniformly convex Banach space with Opial’s condition. Finally, we show that if \( E \) is uniform convex and if the ranges of \( T \) are contained in a compact subset of \( C \), the iterates \( \{x_n\} \) defined by (1.2) converge strongly to a fixed point of \( T \). This paper also improves recent theorems due to Sharma and Sahu [9] using ideas of Takahashi-Kim [12].

2. Preliminaries. Throughout this paper, we denote by \( E \) and \( E^* \) a real Banach space and the dual space of \( E \), respectively. The value of \( x^* \in E^* \) at \( x \in E \) is denoted by \( \langle x,x^* \rangle \). Let \( C \) be a nonempty closed convex subset of \( E \) and let \( T \) be a mapping from \( C \) into itself. Then we denote by \( F(T) \) the set of all fixed points of \( T \), i.e., \( F(T) = \{x \in C : Tx = x \} \). We also denote by \( \mathbb{N} \) the set of all natural numbers and by \( \mathbb{R} \) and \( \mathbb{R}^+ \) the sets of all real numbers and all nonnegative real numbers, respectively.

A Banach space \( E \) is called uniformly convex if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for \( x,y \in E \) with \( \|x\|,\|y\| \leq 1 \) and \( \|x-y\| \geq \varepsilon \), \( \|x+y\| \leq 2(1-\delta) \) holds. When \( \{x_n\} \) is a sequence in \( E \), then \( x_n \to x \) (resp., \( x_n \to x \), \( x_n \to x \)) denote strong (resp., weak, weak\(^*\)) convergence of the sequence \( \{x_n\} \) to \( x \). A Banach space \( E \) is said to satisfy Opial’s condition [7] if for any sequence \( \{x_n\} \) in \( E \), \( x_n \to x \) implies that

\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \quad \forall y \in E \text{ with } y \neq x. \tag{2.1}
\]

If \( I-T \) is demiclosed at zero [1], i.e., for any sequence \( \{x_n\} \) in \( C \), the conditions \( x_n \to x \) weakly and \( x_n - Tx_n \to 0 \) strongly imply \( x - Tx = 0 \). With each \( x \in E \), we associate the set

\[
J_\phi(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\| \text{ and } \|x^*\| = \phi(\|x\|) \}, \tag{2.2}
\]

where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous and strictly increasing function with \( \phi(0) = 0 \) and \( \phi(\infty) = \infty \). Then \( J_\phi : E \to 2^{E^*} \) is said to be the duality mapping. Suppose that \( J_\phi \) is single-valued. Then \( J_\phi \) is said to be weakly sequentially continuous if for each \( \{x_n\} \in E \) with \( x_n \to x \), then \( J_\phi(x_n) \xrightarrow{w} J_\phi(x) \). For abbreviation, we set \( J := J_\phi \). In all our proofs we assume, without loss of generality, that \( J \) is normalized. We know that if \( E \) admits a weakly sequentially continuous duality mapping, then \( E \) satisfies Opial’s condition; see [4]. Let \( S(E) = \{x \in E : \|x\| = 1\} \). Then the norm of \( E \) is said to be Gâteaux differentiable (and \( E \) is said to be smooth) if

\[
\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.3}
\]

exists for each \( x \) and \( y \) in \( S(E) \). It is also said to be Fréchet differentiable if, for each \( x \in S(E) \), the limit (2.3) is attained uniformly in \( y \in S(E) \). All Hilbert spaces and
$l^p$ ($1 < p < \infty$) satisfy Opial’s condition, while $L^p$ with $1 < p \neq 2 < \infty$ do not. It is well known that if $E$ is smooth, then the duality mapping $J$ is single-valued and strong-weak* continuous; for more details, see [2] or [11].

3. Convergence theorems. We first begin with the following.

**Lemma 3.1** (see [8]). Let $E$ be a uniformly convex Banach space, $0 < b \leq t_n \leq c < 1$ for all $n \geq 1$, and $a \geq 0$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences of $E$ such that $\limsup_{n \to \infty} \|x_n\| \leq a$, $\limsup_{n \to \infty} \|y_n\| \leq a$, and $\limsup_{n \to \infty} \|t_n x_n + (1-t_n) y_n\| = a$. Then $\limsup_{n \to \infty} \|x_n - y_n\| = 0$.

Using Lemma 3.1, we have the following.

**Lemma 3.2.** Let $C = \bigcup_{i=1}^{n} C_i$ be a union of nonempty closed convex subsets $C_i$ of a uniformly convex Banach space $E$ and let $T,S : C \to C$ be $\lambda$-firmly nonexpansive for some $\lambda \in (0,1)$ and $tT(sTx + (1-s)x) + (1-t)Sx \in C$ for all $x \in C$ and $s,t \in (0,1)$. Then $F(T) \cap F(S)$ is nonempty if and only if the iterates $\{x_n\}$ defined by (1.3) is bounded, $\{x_n - T x_n\}$ and $\{x_n - S x_n\}$ converge strongly to zero as $n \to \infty$.

**Proof.** Let $w$ be a common fixed point of $T$ and $S$. Since $T$ and $S$ are $\lambda$-firmly nonexpansive for some $\lambda \in (0,1)$, it is easy to check that $\|x_{n+1} - w\| \leq \|x_n - w\|$ for all $n \geq 1$. So, $\{x_n\}$ is bounded and $\liminf_{n \to \infty} \|x_n - w\|$ exists. Put $c = \liminf_{n \to \infty} \|x_n - w\|$. Since $T$ is $\lambda$-firmly nonexpansive for some $\lambda \in (0,1)$, we obtain

$$
\|T y_n - w\| \leq \|(1 - \lambda)(y_n - w) + \lambda(T y_n - w)\|
\leq (1 - \lambda)\|y_n - w\| + \lambda\|T y_n - w\|,
$$

(3.1)

and thus $\|T y_n - w\| \leq \|y_n - w\|$. Taking $\limsup_{n \to \infty}$ in both sides, we obtain

$$
\limsup_{n \to \infty} \|T y_n - w\| \leq \limsup_{n \to \infty} \|y_n - w\| \leq \limsup_{n \to \infty} \|x_n - w\| = c.
$$

(3.2)

Furthermore, since

$$
\lim_{n \to \infty} \|\alpha_n (T y_n - w) + (1 - \alpha_n) (S x_n - w)\| = \lim_{n \to \infty} \|x_{n+1} - w\| = c,
$$

(3.3)

by Lemma 3.1, we have $\lim_{n \to \infty} \|T y_n - S x_n\| = 0$. Since

$$
\|x_{n+1} - w\| \leq \alpha_n \|T y_n - w\| + (1 - \alpha_n)\|x_n - w\|
\leq \alpha_n \|y_n - w\| + (1 - \alpha_n)\|x_n - w\|,
$$

(3.4)

we have

$$
\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \leq \|y_n - w\| - \|x_n - w\|.
$$

(3.5)

Since $\{\alpha_n\}$ is assumed to be bounded away from zero, we obtain

$$
c \leq \liminf_{n \to \infty} \|y_n - w\|.
$$

(3.6)

Since $\|y_n - w\| \leq \|x_n - w\|$ for all $n \geq 1$, we have

$$
c = \lim_{n \to \infty} \|y_n - w\| = \lim_{n \to \infty} \|\beta_n (T x_n - w) + (1 - \beta_n)(x_n - w)\|.
$$

(3.7)
By Lemma 3.1, we have $\lim_{n \to \infty} \|Tx_n - x_n\| = 0$. Since
\[
\|x_n - Sx_n\| \leq \|x_n - Tx_n\| + \|Tx_n - Ty_n\| + \|Ty_n - Sx_n\|
\leq (1 + \beta_n)\|x_n - Tx_n\| + \|Ty_n - Sx_n\|,
\]
we have $x_n - Sx_n \to 0$ as $n \to \infty$.

Conversely, suppose that $\{x_n\}$ is bounded, $\{x_n - Tx_n\}$ and $\{x_n - Sx_n\}$ converge strongly to zero as $n \to \infty$. Then we can consider a real-valued function $g$ on $C$ given by
\[
g(v) = \limsup_{n \to \infty} \|x_n - v\| \quad \text{for each } v \in C.
\]

By [11], we know that $g : C \to \mathbb{R}$ is continuous and convex. Further, if $\|v_n\| \to \infty$, then $g(v_n) \to \infty$. So, we have an element $v_0 \in C$ such that $g(v_0) = r = \min_{v \in C} g(v)$. Set $M = \{v_0 \in C : r = g(v_0)\}$. Then $M$ is bounded, closed, and convex. Further, $M$ is invariant under $T$. In fact, let $z \in M$. Then, for some $\lambda \in (0,1)$, we have
\[
\limsup_{n \to \infty} \|Tx_n - Tz\| \leq \limsup_{n \to \infty} \|(1 - \lambda)(x_n - z) + \lambda(Tx_n - Tz)\|
\leq (1 - \lambda)\limsup_{n \to \infty} \|x_n - z\| + \lambda\limsup_{n \to \infty} \|Tx_n - Tz\|
\]
and thus
\[
\limsup_{n \to \infty} \|x_n - Tz\| = \limsup_{n \to \infty} \|Tx_n - Tz\| \leq \limsup_{n \to \infty} \|x_n - z\|.
\]

Hence $Tz \in M$. Similarly, $M$ is invariant under $S$. Since $E$ is uniformly convex and hence $M$ consists of one point, we have a common fixed point of $T$ and $S$ in $M$; see [13].

**Remark 3.3.** In Lemma 3.2, if $F(T) \cap F(S) \neq \emptyset$, then we furthermore see that $\{y_n - Ty_n\}$ and $\{y_n - Sy_n\}$ converge strongly to zero as $n \to \infty$.

We first consider the following weak convergence of $\lambda$-firmly nonexpansive mappings in a Banach space.

**Theorem 3.4.** Let $E$ be a uniformly convex Banach space satisfying Opial’s condition and let $C = \bigcup_{i=1}^{n} C_i$ be a union of nonempty closed convex subsets $C_i$ of $E$ and let $T, S : C \to C$ be $\lambda$-firmly nonexpansive for some $\lambda \in (0,1)$ with a common fixed point and $tT(sTx + (1 - s)x) + (1 - t)Sx \in C$ for all $x \in C$ and $s, t \in (0,1)$. Then the iterates $\{x_n\}$ and $\{y_n\}$ defined by (1.3) converge weakly to a common fixed point of $T$ and $S$. Further, the two $w$-limits of $\{x_n\}$ and $\{y_n\}$ coincide.

**Proof.** Let $z$ be a common fixed point of $T$ and $S$. Then, as in the proof of Lemma 3.2, we have $\lim_{n \to \infty} \|x_n - z\|$ exists. Let $z_1$ and $z_2$ be two weak subsequential limits of the sequence $\{x_n\}$. We claim that the conditions $x_{n_i} - z_1$ and $x_{n_j} - z_2$ imply $z_1 = z_2 \in F(T) \cap F(S)$. We first show that $z_1, z_2 \in F(T)$. In fact, if $Tz_1 \neq z_1$, then, by Opial’s condition, we have $\limsup_{i \to \infty} \|x_{n_i} - z_1\| < \limsup_{i \to \infty} \|x_{n_i} - Tz_1\|$. Since $T$ is $\lambda$-firmly nonexpansive for some $\lambda \in (0,1)$, we obtain
\[
\limsup_{i \to \infty} \|Tx_{n_i} - Tz_1\| \leq \limsup_{i \to \infty} \|(1 - \lambda)(x_{n_i} - z_1) + \lambda(Tx_{n_i} - Tz_1)\|
\leq (1 - \lambda)\limsup_{i \to \infty} \|x_{n_i} - z_1\| + \lambda\limsup_{i \to \infty} \|Tx_{n_i} - Tz_1\|.
\]
By Lemma 3.2, we have
\[
\limsup_{i \to \infty} \|x_{n_i} - Tz_1\| = \limsup_{i \to \infty} \|x_{n_i} - z_1\|. \tag{3.13}
\]
This is a contradiction. Hence we have \(Tz_1 = z_1\). Similarly, we have \(z_2 \in F(T)\). Next, we show \(z_1 = z_2\). If not, by Opial’s condition,
\[
\lim_{n \to \infty} \|x_n - z_1\| = \lim_{n \to \infty} \|x_{n_i} - z_1\| < \lim_{j \to \infty} \|x_{n_j} - z_2\|
\]
\[
= \lim_{j \to \infty} \|x_n - z_2\| = \lim_{j \to \infty} \|x_{n_j} - z_2\|
\]
\[
< \lim_{j \to \infty} \|x_{n_j} - z_1\| = \lim_{n \to \infty} \|x_n - z_1\|. \tag{3.14}
\]
This is a contradiction. Hence we have \(z_1 = z_2\). By using the same method as above, we have \(z_1 = z_2 \in F(S)\). This implies that \(\{x_n\}\) converges weakly to a common fixed point of \(T\) and \(S\). As in the proof of Lemma 3.2, we have \(\lim_{n \to \infty} \|y_n - z\|\) exists. Let \(y_{n_i} \to w_1\) and \(y_{n_j} \to w_2\). Then, by using the same method as above, we obtain \(w_1 = w_2 \in F(T) \cap F(S)\). Further, since \(\|x_n - y_n\| = \beta_n \|x_n - Tx_n\| \to 0\) as \(n \to \infty\), we readily see that the two \(w\)-limits of \(\{x_n\}\) and \(\{y_n\}\) coincide.

**Theorem 3.5.** Let \(E\) be a uniformly convex Banach space with a Fréchet differentiable norm. Let \(C = \bigcup_{i=1}^{n} C_i\) be a union of nonempty closed convex subsets \(C_i\) of \(E\) and let \(T, S : C \to C\) be \(\lambda\)-firmly nonexpansive for some \(\lambda \in (0, 1)\) with a common fixed point, and let \(I - T, I - S\) be demiclosed at zero and \(tT(sT x + (1 - s)x) + (1 - t)Sx \in C\) for all \(x \in C\) and \(s, t \in (0, 1)\). Then the iterates \(\{x_n\}\) and \(\{y_n\}\) defined by (1.3) converge weakly to a common fixed point of \(T\) and \(S\). Further, the two \(w\)-limits of \(\{x_n\}\) and \(\{y_n\}\) coincide.

**Proof.** Since \(F(T) \cap F(S)\) is nonempty, it follows from Lemma 3.2 that \(\{x_n\}\) is bounded, \(\{x_n - Tx_n\}\) and \(\{x_n - Sx_n\}\) converge strongly to zero as \(n \to \infty\). There exists a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) and a point \(z \in C\) such that \(x_{n_i} \to z\). Since \(I - T\) and \(I - S\) are demiclosed at zero, we obtain \(z \in F(T) \cap F(S)\). For \(y, z \in F(T) \cap F(S)\), as in the proof of Lemma 2 [12], we have \(\lim_{n \to \infty} \langle x_n, J(z - y) \rangle \) exists. To prove Theorem 3.5, assume \(x_{n_i} \to z_1\) and \(x_{n_j} \to z_2\). Then, for \(y, z \in F(T) \cap F(S)\), we have
\[
\langle z_1, J(y - z) \rangle = \lim_{i \to \infty} \langle x_{n_i}, J(y - z) \rangle = \lim_{n \to \infty} \langle x_n, J(y - z) \rangle
\]
\[
= \lim_{j \to \infty} \langle x_{n_j}, J(y - z) \rangle = \langle z_2, J(y - z) \rangle. \tag{3.15}
\]
Setting \(y = z_1\) and \(z = z_2\), we obtain \(\langle z_1 - z_2, J(z_1 - z_2) \rangle = 0\) and hence \(z_1 = z_2\). This implies that \(\{x_n\}\) converges weakly to a common fixed point of \(T\) and \(S\). By using the same method as above, \(\{y_n\}\) converges weakly to a common fixed point of \(T\) and \(S\). Further, since \(x_n - y_n \to 0\) as \(n \to \infty\), the remaining part of the proof is trivial.

**Theorem 3.6.** Let \(E\) be a uniformly convex Banach space satisfying Opial’s condition, and let \(C = \bigcup_{i=1}^{n} C_i\) be a union of nonempty bounded closed convex subsets \(C_i\) of \(E\) and let \(T : C \to C\) be \(\lambda\)-firmly nonexpansive for some \(\lambda \in (0, 1)\) and \(tT(sT x + (1 - s)x) + (1 - t)x \in C\) for all \(x \in C\) and \(s, t \in (0, 1)\). Then for any initial data \(x_1 \in C\), the iterates \(\{x_n\}\) defined by (1.2), where \(\{\alpha_n\}\) and \(\{\beta_n\}\) are chosen so that \(\alpha_n \in [a, b]\) and
\[ \beta_n \in [0, b] \text{ or } \alpha_n \in [a, 1] \text{ and } \beta_n \in [a, b] \text{ for some } a, b \text{ with } 0 < a \leq b < 1, \text{ converge weakly to a fixed point of } T. \]

**Proof.** The existence of a fixed point follows from Smarzewski [10]. Let \( w \) be a fixed point of \( T \). Then, as in the proof of Lemma 3.2, we have \( \lim_{n \to \infty} \|x_n - w\| \) exists. Put \( \alpha = \lim_{n \to \infty} \|x_n - w\| \). Since \( T \) is \( \lambda \)-firmly nonexpansive for some \( \lambda \in (0, 1) \), we obtain

\[
\|Ty_n - w\| \leq \|(1 - \lambda)(y_n - w) + \lambda(Ty_n - w)\|
\leq (1 - \lambda)\|y_n - w\| + \lambda\|Ty_n - w\|
\]

(3.16)

and thus \( \|Ty_n - w\| \leq \|y_n - w\| \). Taking \( \limsup_{n \to \infty} \) in both sides, we obtain

\[
\limsup_{n \to \infty} \|Ty_n - w\| \leq \limsup_{n \to \infty} \|x_n - w\| = \alpha.
\]

(3.17)

Further, we have

\[
\lim_{n \to \infty} \|\alpha_n(Ty_n - w) + (1 - \alpha_n)(x_n - w)\| = \lim_{n \to \infty} \|x_{n+1} - w\| = \alpha.
\]

(3.18)

If \( 0 < a \leq \alpha_n \leq b < 1 \) and \( 0 \leq \beta_n \leq b < 1 \), by Lemma 3.1, we have \( \lim_{n \to \infty} \|Ty_n - x_n\| = 0 \). Since

\[
\|Tx_n - x_n\| \leq \|Tx_n - Ty_n\| + \|Ty_n - x_n\|
\leq \|Tx_n - y_n\| + \|Ty_n - x_n\|
\leq \beta_n(\|Tx_n - x_n\| + \|Ty_n - x_n\|),
\]

we obtain

\[
(1 - b)\|Tx_n - x_n\| \leq (1 - \beta_n)\|Tx_n - x_n\| \leq \|Ty_n - x_n\|.
\]

(3.20)

Therefore \( \|x_n - Tx_n\| \to 0 \) as \( n \to \infty \). On the other hand, we have, for all \( n \geq 1 \),

\[
\|x_{n+1} - w\| \leq \alpha_n\|Ty_n - w\| + (1 - \alpha_n)\|x_n - w\|
\leq \alpha_n\|y_n - w\| + (1 - \alpha_n)\|x_n - w\|
\]

(3.21)

and hence

\[
\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \leq \|y_n - w\| - \|x_n - w\|.
\]

(3.22)

If \( 0 < a \leq \alpha_n \leq 1 \) and \( 0 < a \leq \beta_n \leq b < 1 \), we obtain

\[
c \leq \liminf_{n \to \infty} \|y_n - w\|.
\]

(3.23)

Since \( \|y_n - w\| \leq \|x_n - w\| \) for all \( n \geq 1 \), we obtain

\[
c = \lim_{n \to \infty} \|y_n - w\| = \lim_{n \to \infty} \|\beta_n(Tx_n - w) + (1 - \beta_n)(x_n - w)\|.
\]

(3.24)

By Lemma 3.1, we have \( \lim_{n \to \infty} \|Tx_n - x_n\| = 0 \). As in the proof of Theorem 3.4, the result follows. \( \square \)

**Corollary 3.7.** Let \( E \) be a uniformly convex Banach space possessing a weakly sequentially continuous duality mapping and let \( C = \bigcup_{i=1}^{n} C_i \) be a union of nonempty bounded closed convex subsets \( C_i \) of \( E \) and let \( T : C \to C \) be a \( \lambda \)-firmly nonexpansive for some \( \lambda \in (0, 1) \) and let \( tI - T \) be demiclosed at zero and \( tTx + (1 - t)x \in C \) for all \( x \in C \) and \( t \in (0, 1) \). Let \( \{\alpha_n\} \) be a real sequence satisfying \( 0 < a \leq \alpha_n \leq b < 1 \) for all \( n \in \mathbb{N} \).
Pick \( x_1 \in C \) and define \( x_{n+1} = \alpha_n Tx_n + (1 - \alpha_n)x_n \) for all \( n \in \mathbb{N} \). Then \( \{x_n\} \) converges weakly to a fixed point of \( T \).

**Corollary 3.8.** Let \( E \) be a uniformly convex Banach space possessing a weakly sequentially continuous duality mapping and let \( C = \bigcup_{i=1}^{n} C_i \) be a union of nonempty bounded closed convex subsets \( C_i \) of \( E \) and let \( T : C \rightarrow C \) be \( \lambda \)-firmly nonexpansive for some \( \lambda \in (0, 1) \) and let \( I - T \) be demiclosed at zero and \( tT(sTx + (1 - s)x) + (1 - t)x \in C \) for all \( x \in C \) and \( s, t \in (0, 1) \). Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be two sequence real sequence satisfying \( 0 < a \leq \alpha_n \leq b < 1 \) and \( 0 < c \leq \beta_n \leq d < 1 \) for all \( n \in \mathbb{N} \). Pick \( x_1 \in C \) and the iterates \( \{x_n\} \) defined by (1.2). Then \( \{x_n\} \) converges weakly to a fixed point of \( T \).

Next, we consider a strong convergence of \( \lambda \)-firmly nonexpansive mapping in a Banach space.

**Theorem 3.9.** Let \( E \) be a uniformly convex Banach space and let \( C = \bigcup_{i=1}^{n} C_i \) be a union of nonempty bounded closed convex subsets \( C_i \) of \( E \) with \( C_i \subseteq C_{i+1} \). Suppose that \( T : C \rightarrow C \) is \( \lambda \)-firmly nonexpansive for some \( \lambda \in (0, 1) \) such that \( T(C) \) is contained in a compact subset of \( C \). Then for any initial data \( x_1 \in C \), the iterates \( \{x_n\} \) defined by (1.2), where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are chosen so that \( \alpha_n \in [a, b] \) and \( \beta_n \in [0, b] \) or \( \alpha_n \in [a, 1] \) and \( \beta_n \in [a, b] \) for some \( a, b \) with \( 0 < a \leq b < 1 \), converge strongly to a fixed point of \( T \).

**Proof.** Note that \( \{x_n\} \) is well defined. The existence of a fixed point follows from Smarzewski [10]. By Mazur’s theorem [3], \( \overline{\mathcal{C}}(\{x_1\} \cup T(C)) \) is a compact subset of \( C \) containing \( \{x_n\} \). There exist a subsequence \( \{x_m\} \) of the sequence \( \{x_n\} \) and a point \( z \in C \) such that \( x_m \rightarrow z \). As in the proof of Theorem 3.6, \( \{x_n - Tx_n\} \) converges strongly to zero as \( n \rightarrow \infty \). Since \( T \) is \( \lambda \)-firmly nonexpansive for some \( \lambda \in (0, 1) \), we obtain

\[
\|z - Tz\| \leq \|z - x_m\| + \|x_m - Tx_m\| + \|Tx_m - Tz\| \\
\leq 2\|z - x_m\| + \|x_m - Tx_m\| \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty.
\]

Hence \( Tz = z \). As in the proof of Lemma 3.2, we have \( \lim_{n \rightarrow \infty} \|x_n - z\| \) exists. Hence we have \( \lim_{n \rightarrow \infty} \|x_n - z\| = 0 \).

**Remark 3.10.** In Theorem 3.9, if \( T, S : C \rightarrow C \) are \( \lambda \)-firmly nonexpansive for some \( \lambda \in (0, 1) \) such that \( T(C) \) and \( S(C) \) are contained in a compact subset of \( C \) and \( F(T) \cap F(S) \neq \emptyset \), then the iterates \( \{x_n\} \) and \( \{y_n\} \) defined by (1.3) converge strongly to the same common fixed point of \( T \) and \( S \).

**References**


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