MATRIX-VARIATE BETA DISTRIBUTION

ARJUN K. GUPTA and DAYA K. NAGAR

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ABSTRACT. We propose matrix-variate beta type III distribution. Several properties of this distribution including Laplace transform, marginal distribution and its relationship with matrix-variate beta type I and type II distributions are also studied.

Keywords and phrases. Matrix variate, beta distribution, zonal polynomial, transformation, Laplace transform.

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1. Introduction. The random variable \( u \) with the probability density function (pdf)
\[
(\beta(a,b))^{-1} u^{a-1} (1-u)^{b-1}, \quad 0 < u < 1,
\]
where \( a > 0 \) and \( b > 0 \), is said to have a beta type I distribution with parameters \((a,b)\). The random variable \( v \) with pdf,
\[
(\beta(a,b))^{-1} v^{a-1} (1+v)^{-(a+b)}, \quad v > 0,
\]
where \( a > 0 \) and \( b > 0 \), is said to have beta type II distribution with parameters \((a,b)\). Since (1.2) can be obtained from (1.1) by the transformation \( v = u/(1-u) \), some authors call the distribution of \( v \) an “inverted beta distribution.”

The matrix variate generalizations of (1.1) and (1.2) are given as follows (see [1, 3, 4, 6, 11]).

DEFINITION 1.1. A \( p \times p \) random symmetric positive definite matrix \( U \) is said to have a matrix-variate beta type I distribution with parameters \((a,b)\), denoted as \( U \sim B_p(a,b) \), if its pdf is given by
\[
(\beta_p(a,b))^{-1} \det(U)^{a-(p+1)/2} \det(I_p-U)^{b-(p+1)/2}, \quad 0 < U < I_p,
\]
where \( a > (p-1)/2 \), \( b > (p-1)/2 \), and \( \beta_p(a,b) \) is the multivariate beta function given by
\[
\beta_p(a,b) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)},
\]
where
\[
\Gamma_p(a) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma \left( a - \frac{j-1}{2} \right), \quad \text{Re}(a) > \frac{p-1}{2}.
\]
A. K. GUPTA AND D. K. NAGAR

1.2. A $p \times p$ random symmetric positive definite matrix $V$ is said to have a matrix-variate beta type II distribution with parameters $(a, b)$, denoted as $V \sim B_{II}^{p}(a, b)$, if its pdf is given by

$$
(\beta_{p}(a, b))^{-1} \frac{\det(V)^{(a-\frac{p+1}{2})} \det((I_p + V)^{-(a+b)})}{\det(I_p + V)^{-(a+b)}}, \quad V > 0,
$$

(1.6)

where $a > (p-1)/2, b > (p-1)/2$, and $\beta_{p}(a, b)$ is the multivariate beta function.

As in the univariate case, the density (1.6) can be obtained from (1.3) by transforming $U = (I_p + V)^{-1} V$, together with the Jacobian $J(U \rightarrow V) = \det((I_p + V)^{-1})$.

The matrix-variate beta type I and type II distributions have been studied by many authors, e.g., see [7, 9, 10, 13, 14].

In this paper, a new matrix-variate beta distribution has been defined. We call it “Matrix-variate beta type III” distribution, which is then derived by using matrix transformation. Several properties of this distribution and its relationship with matrix-variate beta type I and type II distributions have also been studied.

2. Density function. First, we define the matrix-variate beta distribution of type III.

Definition 2.1. A $p \times p$ random symmetric positive definite matrix $W$ is said to have a matrix-variate beta type III distribution with parameters $(a, b)$, denoted as $W \sim B_{III}^{p}(a, b)$, if its pdf is given by

$$
2^b \beta_{p}(a, b)^{-1} \frac{\det(W)^{b-(p+1)/2} \det((I_p - W)^{a-(p+1)/2} \det((I_p + W)^{-(a+b)}), \quad 0 < W < I_p,
$$

(2.1)

where $a > (p-1)/2, b > (p-1)/2$, and $\beta_{p}(a, b)$ is the multivariate beta function.

For $p = 1$, the beta type III density is given by

$$
2^b \beta(a, b)^{-1} w^{b-1}(1-w)^{a-1}(1+w)^{-(a+b)}, \quad 0 < w < 1,
$$

(2.2)

and in this case we write $w \sim B_{III}(a, b)$.

By means of a bilinear transformation of the random matrix $U$, the matrix-variate beta type III distribution is generated as in the following theorem.

Theorem 2.1. Let $U \sim B_{I}^{p}(a, b).$ Define $W = (I_p + U)^{-1}(I_p - U).$ Then $W \sim B_{III}^{p}(a, b)$.

Proof. Making the transformation $W = (I_p + U)^{-1}(I_p - U)$ with the Jacobian [12] in the pdf (1.3) of $U$, $J(U \rightarrow W) = 2^{p(p+1)/2} \det((I_p + W)^{-(p+1)})$, we get the desired result.

From the density of beta type III matrix it is apparent that

$$
\int_{0 < W < I_p} \det(W)^{b-(p+1)/2} \det((I_p - W)^{a-(p+1)/2} \det((I_p + W)^{-(a+b)} dW = 2^{-p} \beta_{p}(a, b).
$$

(2.3)
The cumulative distribution function (cdf) of $W$ is obtained as

$$P(W < \Omega) = 2^{p_b} (\beta_p(a, b))^{-1} \int_{0 < W < \Omega} \det(W)^{b-(p+1)/2} \det(I_p-W)^{a-(p+1)/2} \times \det(I_p+W)^{-(a+b)} dW. \quad (2.4)$$

For evaluating the above integral, we use the following results involving zonal polynomials [2, 5, 8]:

$$\det(I_p-W)^{a-(p+1)/2} = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-a+(p+1)/2)_k}{k!} C_\kappa(W), \quad (2.5)$$

$$\det(I_p+W)^{-(a+b)} = \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a+b)_\tau}{t!} (-1)^t C_\tau(W), \quad (2.6)$$

$$C_\kappa(W) C_\tau(W) = \sum_{\delta} g_{\kappa,\tau}^\delta C_\delta(W), \quad (2.7)$$

$$\int_{0 < W < I_p} \det(W)^{a-(p+1)/2} \det(I_p-W)^{b-(p+1)/2} C_\delta(RW) dW$$

$$= \beta_p(a, b) \frac{(a)_\kappa}{(a+b)_\kappa} C_\kappa(R), \quad \text{Re}(a) > \frac{p-1}{2}, \ \text{Re}(b) > \frac{p-1}{2}, \quad (2.8)$$

where $\kappa = (k_1, \ldots, k_p)$, $k_1 \geq \cdots \geq k_p \geq 0$, $k_1 + \cdots + k_p = k$, $\tau = (t_1, \ldots, t_p)$, $t_1 \geq \cdots \geq t_p \geq 0$, $t_1 + \cdots + t_p = t$, $\delta = (d_1, \ldots, d_p)$, $d_1 \geq \cdots \geq d_p \geq 0$, $d_1 + \cdots + d_p = d = k + t$, and $g_{\kappa,\tau}^\delta$ is the coefficient of $C_\delta(W)$ in $C_\kappa(W) C_\tau(W)$. Substituting (2.5) and (2.6) in (2.4) and subsequently using (2.7), we obtain

$$P(W < \Omega) = 2^{p_b} (\beta_p(a, b))^{-1} \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{\tau} \frac{(-a+(p+1)/2)_k}{k!} \frac{(a+b)_\tau}{t!} (-1)^t \sum_{\delta} g_{\kappa,\tau}^\delta \int_{0 < W < \Omega} \det(W)^{b-(p+1)/2} C_\delta(W) dW. \quad (2.9)$$

Now substituting $X = \Omega^{-1/2} W \Omega^{-1/2}$ with the Jacobian $J(W \to X) = \det(\Omega)^{(p+1)/2}$ in the above integral, we get

$$P(W < \Omega) = 2^{p_b} (\beta_p(a, b))^{-1} \det(\Omega)^{b} \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{\tau} \frac{(-a+(p+1)/2)_k}{k!} \frac{1}{t!} (-1)^t \sum_{\delta} g_{\kappa,\tau}^\delta \int_{0 < X < \Omega_p} \det(X)^{b-(p+1)/2} C_\delta(\Omega X) dX$$

$$= \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a+b)_\tau}{t!} (-1)^t \sum_{\delta} g_{\kappa,\tau}^\delta \frac{(b)_\delta}{(b+(p+1)/2)_\delta} C_\delta(\Omega), \quad (2.10)$$

where the last step has been obtained by using (2.8).
The Laplace transform of the density of $W$ is

$$L(Z) = 2^{pb} (\beta_p(a, b))^{-1} \int_{0 \leq W < I_p} \text{etr}(-ZW) \det(W)^{b-(p+1)/2} \times \det(I_p - W)^{a-(p+1)/2} \det(I_p + W)^{-(a+b)} dW,$$

where $Z(p \times p) = ((1 + \delta_{ij})Z_{ij}/2).$ Now, using the expansions

$$
\begin{align*}
\det(I_p + W)^{-(a+b)} &= 2^{-p(a+b)} \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a+b)_\tau}{t!} \left(\frac{1}{2}\right)^t C_\tau(I_p - W), \\
\text{etr}(-ZW) &= \text{etr}(-Z) \text{etr}[Z(I_p - W)] = \text{etr}(-Z) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} C_\kappa(Z(I_p - W)),
\end{align*}
$$

in (2.11), we obtain

$$L(Z) = 2^{-pa} (\beta_p(a, b))^{-1} \text{etr}(-Z) \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\tau} \frac{1}{k!} \frac{(a+b)_\tau}{2^t t!} \Phi(Z),$$

where

$$
\Phi(Z) = \int_{0 \leq W < I_p} \det(W)^{b-(p+1)/2} \det(I_p - W)^{a-(p+1)/2} C_\kappa(Z(I_p - W)) C_\tau(I_p - W) dW \\
= \int_{0 \leq X < I_p} \det(I_p - X)^{b-(p+1)/2} \det(X)^{a-(p+1)/2} C_\kappa(X) C_\tau(X) dX.
$$

Since $\Phi(Z) = \Phi(H'ZH),$ $H \in O(p),$ integrating out $H$ in $\Phi(H'ZH)$ using [5, equation 23], we have

$$\Phi(Z) = \frac{C_\kappa(Z)}{C_\kappa(I_p)} \int_{0 < X < I_p} \det(I_p - X)^{b-(p+1)/2} \det(X)^{a-(p+1)/2} C_\kappa(X) C_\tau(X) dX.$$

Now using (2.7), and integrating $X$ using (2.8), we obtain

$$
\Phi(Z) = \frac{C_\kappa(Z)}{C_\kappa(I_p)} \sum_\delta \frac{(a)_{\delta}}{(a+b)_{\delta}} \frac{\Gamma_p(a) \Gamma_p(b)}{\Gamma_p(a+b)} C_\delta(I_p).
$$

Substituting (2.16) in (2.13), we finally obtain

$$L(Z) = 2^{-pa} \text{etr}(-Z) \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\tau} \frac{1}{k!} \frac{(a+b)_\tau}{2^t t!} \frac{C_\kappa(Z)}{C_\kappa(I_p)} \sum_\delta \frac{(a)_{\delta}}{(a+b)_{\delta}} C_\delta(I_p).$$

3. Properties. In this section, we study some properties of the random matrix distributed as matrix-variate beta type III.
The density of a random orthogonal matrix, whose elements are either constants or random variables distributed independently of X, is invariant under the transformation $W \rightarrow HWH'$, and does not depend on H.

The relationship between beta type I, type II, and type III matrices is now exhibited. First, we derive the density of $B_p^{III}(a,b)$ in the pdf (2.1) of $W$, the density of $X$ is obtained as

$$2^{p^b} (\beta_p(a,b))^{-1} \det(\beta_p(a,b))^{-1/2} \det(X)^{b-(p+1)/2} \times \det(\beta_p(a,b) + X)^{-(a+b)}, \quad 0 < X < \beta_p(a,b). \quad (3.1)$$

which is the desired result.

We will write $X \sim B_p^{III}(a,b;AA')$. In the next theorem, it is shown that the matrix-variate beta distribution of type III is orthogonally invariant.

**Theorem 3.2.** Let $W \sim B_p^{III}(a,b)$ and $H(p \times p)$ be an orthogonal matrix, whose elements are either constants or random variables distributed independently of $W$. Then, the distribution of $W$ is invariant under the transformation $W \rightarrow HWH'$, and does not depend on $H$.

**Proof.** First, let $H$ be a constant orthogonal matrix. Then, from Theorem 3.1, $HWH' \sim B_p^{III}(a,b)$ since $HH' = I_p$. If, however, $H$ is a random orthogonal matrix, then the conditional distribution of $HH' | H \sim B_p^{III}(a,b)$. Since this distribution does not depend on $H$, $HWH' \sim B_p^{III}(a,b)$.

The relationship between beta type I, type II, and type III matrices is now exhibited. First, we derive the density of $W^{-1}$.

**Theorem 3.3.** Let $W \sim B_p^{III}(a,b)$. Then the density of $Y = W^{-1}$ is

$$2^{p^b} (\beta_p(a,b))^{-1} \det(Y - I_p)^{a-(p+1)/2} \det(I_p + Y)^{-(a+b)}, \quad Y > I_p. \quad (3.3)$$

**Proof.** Making the transformation $Y = W^{-1}$ with the Jacobian $J(W \rightarrow Y) = \det(Y)^{-(p+1)}$, in the density of $W$ the result follows.

The density derived above may be called the inverse beta type III. From Theorem 3.3, it is clear that if $W \sim B_p^{III}(a,b)$, then $W^{-1}$ does not follow beta type III distribution.

**Theorem 3.4.** (i) Let $U \sim B_p^{III}(a,b)$ and $W = (I_p + U)^{-1}(I_p - U)$, then $W \sim B_p^{III}(a,b)$.

(ii) Similarly, if $W \sim B_p^{III}(a,b)$ and $U = (I_p + W)^{-1}(I_p - W)$, then $U \sim B_p^{III}(a,b)$.

**Proof.** Part (i) is proved in Theorem 2.1. Since $W = (I_p + U)^{-1}(I_p - U)$, the Jacobian of transformation is $J(W \rightarrow U) = 2^{p(p+1)/2} \det(I_p + U)^{-(p+1)}$. Now, making the substitution in the density of $W$ given by (2.1) the result (ii) follows.

**Theorem 3.5.** (i) Let $V \sim B_p^{III}(a,b)$ and $W = (I_p + 2V)^{-1}$, then $W \sim B_p^{III}(a,b)$.

(ii) Similarly, if $W \sim B_p^{III}(a,b)$ and $V = (I_p - W)W^{-1}/2$, then $V \sim B_p^{III}(a,b)$.

The marginal distributions of $W$ is given in the following.
THEOREM 3.6. Let \( W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \), \( W_{11} (q \times q) \), and \( W_{22-1} = W_{22} - W_{21}W_{11}^{-1}W_{12} \). If \( W \sim B_p^{\text{ill}}(a, b) \), then \( W_{22-1} \sim B_p^{\text{ill}}(a, b - q/2) \).

PROOF. From the partition of \( W \), we have

\[
\det(W) = \det(W_{11}) \det(W_{22-1}),
\]
(3.4)
\[
\det(I_p - W) = \det(I_q - W_{11}) \det((I_p - q - W_{22-1} - W_{21}W_{11}^{-1}(I_q - W_{11})^{-1}W_{12})) ,
\]
(3.5)
\[
\det(I_p + W) = \det(I_q + W_{11}) \det((I_p - q + W_{22-1} + W_{21}W_{11}^{-1}(I_q + W_{11})^{-1}W_{12}).
\]
(3.6)

Now making the transformation \( W_{11} = W_{11}, X = W_{21}W_{11}^{-1/2}, \) and \( W_{22-1} = W_{22} - W_{21}W_{11}^{-1}W_{12} = W_{22} - XX' \) with Jacobian \( J(W_{11}, W_{22}, W_{21} - W_{11}, W_{22-1}, X) = \det(W_{11})^{(p-q)/2} \) and substituting (3.4), (3.5), and (3.6) in the density of \( W \), we get the joint density of \( W_{11}, W_{22-1}, \) and \( X \) as follows:

\[
2^{p^b}(\beta_p(a, b)^{-1}) \det(W_{11})^{-b(q+1)/2} \det(I_q - W_{11})^{a-(p+1)/2} \det(I_q + W_{11})^{-(a+b)}
\]
\[
\times \det(W_{22-1})^{b-(p+1)/2} \det(I_p - q - W_{22-1})^{a-(p+1)/2} \det(I_p - q + W_{22-1})^{-(a+b)}
\]
\[
\times \det(I_p - q - (I_p - q - W_{22-1})^{-1}X(I_q - W_{11})^{-1}X')^{a-(p+1)/2}
\]
\[
\times \det(I_p - q + (I_p - q + W_{22-1})^{-1}X(I_q + W_{11})^{-1}X')^{-(a+b)}.
\]
(3.7)

Substituting \( Y = (I_p - q - W_{22-1})^{-1/2}X(I_q - W_{11})^{-1/2} \) with Jacobian \( J(X - Y) = \det(I_p - q - W_{22-1})^{q/2} \det(I_q - W_{11})^{(p-q)/2} \) and integrating \( Y \), we get the joint density of \( W_{11} \) and \( W_{22-1} \) as

\[
2^{p^b}(\beta_p(a, b)^{-1}) \det(W_{11})^{-b(q+1)/2} \det(I_q - W_{11})^{a-(q+1)/2} \det(I_q + W_{11})^{-(a+b)}
\]
\[
\times \det(W_{22-1})^{b-(p+1)/2} \det(I_p - q - W_{22-1})^{a-(p-q+1)/2} \det(I_p - q + W_{22-1})^{-(a+b)} g(A, B),
\]
(3.8)

where

\[
g(A, B) = \int_{I_p-q-YY'>0} \det(I_p - q - YY')^{a-(p+1)/2} \det(I_p - q + AYY')^{-(a+b)} dY
\]
\[
= \int_{0 < Z < I_p-q} \int_{YY'=Z} \det(I_p - q - YY')^{a-(p+1)/2} \left( \begin{array}{c}
\frac{1}{2} F_0^{(q)}(a+b; -YY'AB)
\end{array} \right) dY dZ, \text{ for } p - q \leq q,
\]
\[
= \int_{0 < Z < I_p-q} \int_{YY'=Z} \det(I_q - YY')^{a-(p+1)/2} \left( \begin{array}{c}
\frac{1}{2} F_0^{(p-q)}(a+b; -AYY')
\end{array} \right) dY dZ, \text{ for } p - q > q,
\]
(3.9)

with \( A = (I_p - q - W_{22-1})^{1/2}(I_p - q + W_{22-1})^{-1}(I_p - q - W_{22-1})^{1/2} \) and \( B = (I_q - W_{11})^{1/2} (I_q + W_{11})^{-1}(I_q - W_{11})^{1/2} \). Since \( g(A, B) = g(A, H'BH) \), \( H \in O(q) \), by integrating \( H \) in \( g(A, H'BH) \), we obtain
we get the density of type III beta matrices. Let

\[ g(A, B) = \int_{0 < Z < p - q} \int_{Y' = Z} \det \left( I_{p - q} - YY' \right)^{a - (p + 1)/2} F_0^{(q)}(a + b; -AYY', B) dY dZ 
\]

\[ = \frac{\pi^{a(p - q)/2}}{\Gamma_{p - q}(q/2)} \int_{0 < Z < p - q} \det(Z)^{(q - p + q - 1)/2} \det \left( I_{p - q} - Z \right)^{a - (p + 1)/2} F_0^{(q)}(a + b; -AZ, B) dZ 
\]

\[ = \frac{\pi^{a(p - q)/2}}{\Gamma_{p - q}(q/2) \Gamma_{p - q}(a - q/2)} \frac{\Gamma_{p - q}(q/2)}{\Gamma_{p - q}(a)} 2F_1^{(q)} \left( \frac{q}{2}, a + b; -A, B \right) 
\]

\[ = \frac{\pi^{a(p - q)/2} \Gamma_{p - q}(a - q/2)}{\Gamma_{p - q}(a)} 2F_1^{(q)} \left( \frac{q}{2}, a + b; -A, B \right). 
\]

(3.10)

Substituting \( g(A, B) \) in (3.8), we get the joint density of \( W_{11} \) and \( W_{22,1} \) as

\[ 2^{ab} \left( \beta_q(a, b) \right)^{-1} \det(W_{11})^{-b - (q+1)/2} \det(I_{q - W_{11}})^{-a - (q+1)/2} \times \det(I_{q + W_{11}})^{(a+b)2 - (p+1)/2} \]

\[ \times \det(I_{p - q} - W_{11})^{a - (q+1)/2} \det(I_{p - q})^{-a - (q+1)/2} \times 2F_1^{(q)} \left( \frac{q}{2}, a + b; -A, B \right). 
\]

(3.11)

Clearly \( W_{11} \) and \( W_{22,1} \) are not independent. Integrating \( W_{11} \), using

\[ 2^{ab} \int_{0 < W_{11} < I_{q}} \det(W_{11})^{-b - (q+1)/2} \det(I_{q - W_{11}})^{-a - (q+1)/2} \det(I_{q + W_{11}})^{(a+b)2 - (p+1)/2} \]

\[ \times 2F_1^{(q)} \left( \frac{q}{2}, a + b; -A, B \right) dW_{11} 
\]

\[ = \int_{0 < B < I_{q}} \det(B)^{-a - (q+1)/2} \det(I_{q - B})^{-b - (q+1)/2} 2F_1^{(q)} \left( \frac{q}{2}, a + b; -A, B \right) dB 
\]

\[ = \beta_q(a, b) 3F_2^{(q)} \left( a, \frac{q}{2}, a + b; a, a + b; -A \right) 
\]

\[ = \beta_q(a, b) \det(I_{p - q} + A)^{-q/2} 
\]

\[ = 2^{-(p-q)/2} \beta_q(a, b) \det(I_{p - q} + W_{22,1})^{q/2} , 
\]

(3.12)

we get the density of \( W_{22,1} \). For \( q < p - q \), using (3.9) and following similar steps we get the same result.

\( \square \)

Alternately, Theorem 3.6 can be proved using the relationship between type II and type III beta matrices. Let \( V \sim B^I_p(a, b) \) and \( V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \), \( V_{11} (q \times q) \) and \( V_{11:2} = V_{11} - V_{12}V_{22}^{-1}V_{21} \). It is well known that \( V_{11:2} \) and \( V_{22} \) are distributed independently (see [4]), \( V_{11:2} \sim B^I_p(a - (p - q)/2, b) \) and \( V_{22} \sim B^P_{p-q}(a, b - q/2) \). According to Theorem 3.5(i), if \( V \sim B^I_p(a, b) \), then \( W = (I_p + 2V)^{-1} \sim B^P_{p}(a, b) \). Furthermore,

\[ W^{-1} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} I_{q} + 2V_{11} & 2V_{12} \\ 2V_{21} & I_{p - q} + 2V_{22} \end{pmatrix} . 
\]

(3.13)

That is, \( W^{22} ((p - q) \times (p - q)) = W_{22,1}^{-1} = I_{p - q} + 2V_{22} \) and \( W_{22,1} = (I_{q} + 2V_{22})^{-1} \).
Now, since \( V_{22} \sim B_{p-q}^m(a,b-\frac{q}{2}) \), then \( W_{22,1} = (I_{p-q} + 2V_{22})^{-1} \sim B_{p-q}^m(a,b-\frac{q}{2}) \). The distribution of \((AW^{-1}A')^{-1}\) where \( A(q \times p) \) is a constant matrix of rank \( q (\leq p) \), is now derived.

**Theorem 3.7.** Let \( A(q \times p) \) be a constant matrix of rank \( q (\leq p) \). If \( W \sim B_{p}^m(a,b) \), then \((AW^{-1}A')^{-1} \sim B_{q}^m(a,b-(p-q)/2 ; (AA')^{-1})\).

**Proof.** We write \( A = M(I_q \ 0)G \), where \( M(q \times q) \) is nonsingular and \( G(p \times p) \) is orthogonal. Now,

\[
(AW^{-1}A')^{-1} = (M(I_q \ 0)GW^{-1}G'(I_q \ 0)')^{-1}
\]

\[
= (M')^{-1} \left[ (I_q \ 0)Y^{-1} \left( I_q \ 0 \right)' \right]^{-1} M^{-1}
\]

(3.14)

where \( Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = GWG' \sim B_{q}^m(a,b), Y_{11}(q \times q) \) and \( Y_{11} = (Y_{11} - Y_{12}Y_{21}^{-1}Y_{21}) = Y_{11}^{-1} \). From Theorem 3.6, \( Y_{11,2} \sim B_{q}^m(a,b-(p-q)/2) \) and from Theorem 3.1, \((M')^{-1}Y_{22,1}M^{-1} \sim B_{q}^m(a,b-(p-q)/2; (MM')^{-1})\). The proof is now completed by observing that \( MM' = AA' \).

**Corollary 3.8.** Let \( W \sim B_{p}^m(a,b) \) and \( a \in \mathbb{R}^p, a \neq 0 \), then \( a' a/ a' W^{-1} a \sim B_{q}^m(a,b-(p-1)/2) \).

In Corollary 3.8, the distribution of \( a' a/ a' W^{-1} a \) does not depend on \( a \). Thus, if \( y(p \times 1) \) is a random vector, independent of \( W \), and \( P(y \neq 0) = 1 \), then it follows that \( y' y/ y' W^{-1} y \sim B_{q}^m(a,b-(p-1)/2) \).

**Theorem 3.9.** Let \( W \sim B_{p}^m(a,b) \), then

(i) \( \int \frac{\det(W)^h}{\det(I_p + W)^h} = \frac{\Gamma_p(a+b)\Gamma_p(b+h)}{\Gamma_p(b)\Gamma_p(a+b+h)} \cdot \Re(h) > -b + \frac{p-1}{2}, \)

(ii) \( \int \frac{\det(W)^h}{}\frac{\det(I_p - W)^h}{= \frac{2^{-p}\Gamma_p(a+b)\Gamma_p(a+h)}{\Gamma_p(b)\Gamma_p(a+b+h)} \cdot 2F_1(a,a+b;a+b+h,\frac{1}{2}I_p), \ Re(h) > -b + \frac{p-1}{2}, \)

(iii) \( \int \frac{\det(I_p - W)^h}{\Gamma_p(a)\Gamma_p(a+b+h)} \cdot 2F_1(a+h,a+b;a+b+h,\frac{1}{2}I_p), \ Re(h) > -a + \frac{p-1}{2}, \)

where \( 2F_1 \) is the hypergeometric function of matrix argument.

**Proof.** (i) From the density of \( W \), we have

\[
\int \frac{\det(W)^h}{\det(I_p + W)^h} = 2^p (\beta_p(a,b))^{-1} \int_{0 < W < I_p} \det(W)^{b-h - (p+1)/2} \det(I_p - W)^{a-(p+1)/2} \times \det(I_p + W)^{-(a+b+h)} dW
\]

\[
= 2^p (\beta_p(a,b))^{-1} \frac{\beta_p(a+b+h)}{2^{p(b+h)}}, \ Re(h) > -b + \frac{p-1}{2}.
\]

(3.15)
Simplifying this last expression using (1.4), we get the desired result.

(ii) From the density of $W$, we have

$$
E[E(W)^h] = 2^b \beta_p(a, b)^{-1} \int_{0 < W < I_p} \det(W)^{b+\frac{p+1}{2}} \det(I_p-W)^{a-\frac{p+1}{2}} \det(I_p+W)^{-\frac{a+b}{2}} dW.
$$

(3.16)

Writing $\det(I_p + W)^{-\frac{a+b}{2}}$ in series involving zonal polynomials using (2.12), we obtain

$$
E[\det(W)^h] = 2^{-p} a \beta_p(a, b)^{-1} \sum_{l=0}^{\infty} \sum_{\tau} \frac{(a+b)_\tau}{2^l l!} \int_{0 < W < I_p} \det(W)^{b+\frac{p+1}{2}} \det(I_p-W)^{a-\frac{p+1}{2}} C_\tau(I_p) \quad \text{Re}(h) > -b + \frac{p-1}{2},
$$

where the integral has been evaluated using (2.8). Finally, simplifying the expression using results on hypergeometric functions [2, 5], we get the desired result.

Similarly $E[\det(I_p - W)^h]$ can be derived.

From the density of $W$, we have

$$
E[C_k(W)] = 2^b \beta_p(a, b)^{-1} \int_{0 < W < I_p} C_k(W) \det(W)^{b-\frac{p+1}{2}} \det(I_p-W)^{a-\frac{p+1}{2}} \det(I_p+W)^{-\frac{a+b}{2}} dW.
$$

(3.18)

Writing $\det(I_p + W)^{-\frac{a+b}{2}}$ in series involving zonal polynomials using (2.6), we obtain

$$
E[C_k(W)] = 2^b \beta_p(a, b)^{-1} \sum_{l=0}^{\infty} \sum_{\tau} \frac{(a+b)_\tau}{l!} (-1)^t \int_{0 < W < I_p} C_k(W) C_\tau(W) \det(W)^{b-\frac{p+1}{2}} \det(I_p-W)^{a-\frac{p+1}{2}} \det(I_p+W)^{-\frac{a+b}{2}} dW.
$$

(3.19)

where the last two steps have been obtained using (2.7) and (2.8).
Further pdf of \( Y = \det X \), we get the joint density of \( X \) as
\[
2^{p(a+b)} (\beta_p(a,b+c))^{-1} \det(Y)^{a-(p+1)/2} \det(I_p-Y)^{b+c-(p+1)/2} \\
\times \det(I_p+Y)^{-a-b-c} F_1(b,a+b+c; b+c; -(I_p+Y)^{-1}(I_p-Y)), \quad 0 < Y < I_p. 
\] (3.20)

Further, pdf of \( X = (I_p+Y)^{-1}(I_p-Y) = (I_p+U^{1/2}U^{1/2})^{-1}(I_p-I_p)U^{1/2}U^{1/2} \) is
\[
2^b (\beta_p(a,b+c))^{-1} \det(X)^{b+c-(p+1)/2} \det(I_p-X)^{a-(p+1)/2} \\
\times F_1(b,a+b+c; b+c; -X), \quad 0 < X < I_p. 
\] (3.21)

(iii) Let \( U \sim B_p(a,b) \) and \( V \sim B_p(a+b,c) \) be independent. Then \( Y = (I_p + V^{1/2}V^{1/2})^{-1} \)
\( \times (I_p - V^{1/2}V^{1/2}) \sim B_p(a,b+c) \).

**Proof.** The joint density of \( U \) and \( V \) is
\[
2^{p(a+b)} (\beta_p(a,b)\beta_p(c,a+b))^{-1} \det(U)^{a-(p+1)/2} \det(I_p-U)^{b-(p+1)/2} \\
\times \det(W)^{a+b-(p+1)/2} \det(I_p-W)^{c-(p+1)/2} \\
\times \det(I_p+W)^{-(a+b+c)}, \quad 0 < U < I_p, \quad 0 < W < I_p. 
\] (3.22)

Making the transformation \( Y = U^{1/2}U^{1/2} \) with Jacobian \( J(U - Y) = \det(W)^{-(p+1)/2} \)
in above, we get the joint density of \( Y \) and \( W \) as
\[
2^{p(a+b)} (\beta_p(a,b)\beta_p(c,a+b))^{-1} \det(Y)^{a-(p+1)/2} \det(Y)^{b-(p+1)/2} \\
\times \det(I_p-W)^{c-(p+1)/2} \det(I_p+W)^{-(a+b+c)}, \quad 0 < Y < W < I_p. 
\] (3.23)

Substituting \( Z = (I_p-Y)^{1/2}(W-Y)(I_p-Y)^{-1/2} \) with the Jacobian \( J(W - Z) = \det(I_p-Y)^{(p+1)/2} \)
and integrating \( Z \), we get the density of \( Y \) as
\[
2^{p(a+b)} (\beta_p(a,b)\beta_p(c,a+b))^{-1} \det(Y)^{a-(p+1)/2} \\
\times \left. \int_{0<Z<I_p} \det(Z)^{b-(p+1)/2} \det(I_p-Z)^{c-(p+1)/2} \right. \\
\times \det(I_p + (I_p+Y)^{-1}(I_p-Y)Z)^{-(a+b+c)} \, dZ. 
\] (3.24)

Integration of \( Z \) using [5, equation 48] completes the proof.

(ii) From [6], we have \( V^{1/2}V^{1/2} \sim B_p(a,b+c) \) and from Theorem 2.1, we have
\( (I_p + V^{1/2}V^{1/2})^{-1}(I_p - V^{1/2}V^{1/2}) \sim B_p(a,b+c) \).
\( \Box \)
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Arjun K. Gupta: Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, Ohio 43403-0221, USA

Daya K. Nagar: Departamento de Matemáticas, Universidad de Antioquia, Medellín, A.A. 1226, Colombia