A SUBORDINATION THEOREM FOR SPIRALLIKE FUNCTIONS

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Abstract. We prove a subordination relation for a subclass of the class of \( \lambda \)-spirallike functions.

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1. Introduction. Let \( K \) denote the usual class of convex functions. Denote by \( S_p(\lambda) \), \(-\pi/2 < \lambda < \pi/2\), the class of functions \( f(z) = z + az^2 + \cdots \) which are analytic in \( E \) and satisfy therein the condition

\[
\text{Re} \left[ e^{i\lambda} \frac{zf'(z)}{f(z)} \right] > 0. \quad (1.1)
\]

Spacek [3] proved that members of \( S_p(\lambda) \), known as \( \lambda \) spirallike functions, are univalent in \( E \). In 1989, Silverman [2] proved that if

\[
\sum_{n=2}^{\infty} (1 + (n-1) \sec \lambda) |a_n| \leq 1 \quad (|\lambda| < \frac{\pi}{2}), \quad (1.2)
\]

then the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) belongs to \( S_p(\lambda) \). Let us denote by \( G(\lambda) \), the class of function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) whose coefficients satisfy the condition (1.2). Note that \( G(0) \) is a subclass of the class of starlike functions (with respect to the origin) (see Silverman [1]).

In this paper, we prove a subordination theorem for the class \( G(\lambda) \). To state and prove our main result we need the following definitions and lemma.

**Definition 1.1.** If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) are analytic in \( |z| < r \), then their Hadamard product/convolution, \( f \ast g \) is the function defined by the power series

\[
(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \quad (1.3)
\]

The function \( f \ast g \) is also analytic in \( |z| < r \).

**Definition 1.2.** Let \( f \) be analytic in \( E \), \( g \) analytic and univalent in \( E \) and \( f(0) = g(0) \). Then by the symbol \( f(z) \prec g(z) \) (\( f \) subordinate to \( g \)) in \( E \), we shall mean that \( f(E) \subset g(E) \).
**Definition 1.3.** A sequence \( \{b_n\}_1^\infty \) of complex numbers is said to be a subordinating factor sequence if whenever
\[
f(z) = \sum_{k=1}^\infty a_k z^k, \quad a_1 = 1 \text{ is regular, univalent and convex in } E,
\]
we have
\[
\sum_{k=1}^\infty b_k a_k z^k \prec f(z) \text{ in } E. \quad (1.4)
\]

**Lemma 1.4.** The sequence \( \{b_n\}_1^\infty \) is a subordinating factor sequence if and only if
\[
\Re \left[ 1 + 2 \sum_{n=1}^\infty b_n z^n \right] > 0, \quad (z \in E). \quad (1.5)
\]
This lemma which gives a beautiful characterisation of a subordinating factor sequence is due to Wilf [4].

2. Main theorem

**Theorem 2.1.** Let \( f \in G(\lambda) \). Then
\[
\frac{1 + \sec \lambda}{2(2 + \sec \lambda)} (f \ast g)(z) < g(z), \quad (z \in E) \quad (2.1)
\]
for every function \( g \) in the class \( K \).

In particular
\[
\Re f(z) > -\frac{2 + \sec \lambda}{(1 + \sec \lambda)}, \quad (z \in E). \quad (2.2)
\]

The constant \( (1 + \sec \lambda)/2(2 + \sec \lambda) \) cannot be replaced by any larger one.

Taking \( \lambda = 0 \), we obtain the following corollary.

**Corollary 2.2.** If \( f(z) = z + a_2 z^2 + \cdots \) is regular in \( E \) and satisfies therein the condition
\[
\sum_{n=2}^\infty n |a_n| \leq 1, \quad (2.3)
\]
then for every function \( g \) in \( K \), we have
\[
\frac{1}{3} (f \ast g)(z) < g(z), \quad (|z| < 1). \quad (2.4)
\]

In particular, \( \Re f(z) > -3/2 \), \( z \in E \). The constant \( 1/3 \) is best possible.

**Proof of Theorem 2.1.** Let \( f(z) = z + \sum_{n=2}^\infty a_n z^n \) be any member of the class \( G(\lambda) \) and let \( g(z) = z + \sum_{n=2}^\infty c_n z^n \) be any function in the class \( K \). Then
\[
\frac{1 + \sec \lambda}{2(2 + \sec \lambda)} (f \ast g)(z) = \frac{1 + \sec \lambda}{2(2 + \sec \lambda)} \left( z + \sum_{n=2}^\infty a_n c_n z^n \right). \quad (2.5)
\]

Thus, by Definition 1.3, the assertion of our theorem will hold if the sequence
\[
\left( \frac{(1 + \sec \lambda) a_n}{2(2 + \sec \lambda)} \right)_{n=1}^\infty \quad (2.6)
\]
is a subordinating factor sequence, with \( a_1 = 1 \). In view of the lemma, this will be the
case if and only if
\[
\text{Re}\left[1 + 2 \sum_{n=1}^{\infty} \frac{1 + \sec \lambda}{2(2 + \sec \lambda)} a_n z^n\right] > 0, \quad (z \in E).
\] (2.7)

Now
\[
\text{Re}\left[1 + \frac{1 + \sec \lambda}{2 + \sec \lambda} \sum_{n=1}^{\infty} a_n z^n\right]
= \text{Re}\left[1 + \frac{1 + \sec \lambda}{2 + \sec \lambda} z + \frac{1}{2 + \sec \lambda} \sum_{n=2}^{\infty} (1 + \sec \lambda) a_n z^n\right]
\geq \left[1 - \frac{1 + \sec \lambda}{2 + \sec \lambda} r - \frac{1}{2 + \sec \lambda} \sum_{n=2}^{\infty} (1 + (n - 1) \sec \lambda) |a_n| r^n\right]
\geq \left[1 - \frac{1 + \sec \lambda}{2 + \sec \lambda} r - \frac{1}{2 + \sec \lambda} r\right] (|z| = r)
\geq 0.
\] (2.8)

(because \(1 + \sec \lambda \leq 1 + (n - 1) \sec \lambda\) for all \(n \geq 2, |\lambda| < \pi/2\))

Thus (2.7) holds true in \(E\). This proves the first assertion. That \(\text{Re} f(z) > -(2 + \sec \lambda)/(1 + \sec \lambda)\) for \(f \in G(\lambda)\) follows by taking \(g(z) = z/(1 - z)\) in (2.1). To prove the sharpness of the constant \((1 + \sec \lambda)/(2 + \sec \lambda)\), we consider the function \(f_0\) defined by \(f_0(z) = z - (1/(1 + \sec \lambda)) z^2(|\lambda| < \pi/2)\), which is a member of the class \(G(\lambda)\). Thus from the relation (2.1) we obtain
\[
\frac{1 + \sec \lambda}{2(2 + \sec \lambda)} f_0(z) < \frac{z}{1 - z}.
\] (2.9)

It can be easily verified that
\[
\min_{|z| \leq 1} \text{Re}\left[\frac{1 + \sec \lambda}{2(2 + \sec \lambda)} f_0(z)\right] = -\frac{1}{2}.
\] (2.10)

This shows that the constant \((1 + \sec \lambda)/(2 + \sec \lambda)\) is best possible.

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\textbf{References}


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