SMALL BOUND ISOMORPHISMS OF THE DOMAIN OF A CLOSED $\ast$-DERIVATION

TOSHIKO MATSUMOTO and SEIJI WATANABE

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Abstract. The domain $\mathcal{D}(\delta)$ of a closed $\ast$-derivation $\delta$ in $C(K)$ ($K$ : a compact Hausdorff space) is a generalization of the space $C^{(1)}[0,1]$ of differentiable functions on $[0,1]$. In this paper, a problem proposed by Jarosz (1985) is studied in the context of derivations instead of $C^{(1)}[0,1]$.

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Let $K_1$ and $K_2$ be two compact Hausdorff spaces. $C(K_i)$ denotes a space of all complex valued continuous functions on $K_i$ ($i=1,2$). Let $T$ be a surjective linear isometry from $C(K_1)$ to $C(K_2)$. Then the Banach-Stone theorem states that there exist a homeomorphism $\tau$ from $K_2$ to $K_1$ and a function $w$ in $C(K_2)$ with $|w(y)|=1$ ($y\in K_2$) such that

$$Tf(y) = w(y)f(\tau(y)) \quad \text{for } f\in C(K_1), \ y\in K_2.$$  

(1)

That is, the existence of a surjective linear isometry between $C(K_1)$ and $C(K_2)$ implies that $K_1$ and $K_2$ are homeomorphic. Amir [1] and Cambern [2] extended this theorem from this viewpoint as follows.

Theorem 1 (see [1, 2]). If there is a surjective linear isomorphism $T : C(K_1) \to C(K_2)$ such that $\|T\|\|T^{-1}\| < 2$, then $K_1$ and $K_2$ are homeomorphic.

Let $X$ be a compact subset of the real line $\mathbb{R}$ and $C^{(1)}(X)$ be the space of continuously differentiable functions on $X$ with the $\Sigma$-norm defined by $\|f\|_\Sigma = \sup_{x\in X} |f(x)| + \sup_{x\in X} |f'(x)|$.

In [4], Jarosz proposed the following question: “Is there a positive $\varepsilon$ such that for any compact subsets $X,Y$ of the real line $\mathbb{R}$ and any linear isomorphism $T : C^{(1)}(X) \to C^{(1)}(Y)$, $\|T\|\|T^{-1}\| < \varepsilon$ implies that $X$ and $Y$ are homeomorphic?”

In [5], Jun and Lee obtained some partial answers for this question.

Theorem 2 (see [5]). Let $X$ and $Y$ be compact subset of $\mathbb{R}$ and $X \subset [a,b]$ and $Y \subset [c,d]$. If $T$ is a linear isomorphism between $C^{(1)}(X)$ and $C^{(1)}(Y)$ which satisfies

(i) $f'(t) = 0$, then $(Tf)' = 0$,

(ii) $\|fg\| \leq \|TfTg\| \leq (1+\varepsilon)^2 \|fg\|$,

(iii) $\|f\| \leq \|Tf\| \leq (1+\varepsilon)\|f\|$,

(iv) $\varepsilon < \min\{1/49, 1/2(b-a+1), 1/2(c-d+1)\}$,

then $X$ and $Y$ are homeomorphic.
**Theorem 3** [5]. Let $X$ and $Y$ be compact subsets of $\mathbb{R}$ and $X \subset \bigcup_{i=1}^{n}[a_i, b_i]$ $(a_i < b_i < a_{i+1})$ and $\max_i{|b_i - a_i|} < k$ and $Y \subset \bigcup_{j=1}^{m}[c_j, d_j]$ $(c_j < d_j < c_{j+1})$ and $\max_i{|d_j - c_j|} < k$. If $T$ is a linear map from $C^1(X)$ onto $C^1(Y)$ which satisfies

(i) $f'(t) \equiv 0$ if and only if $(Tf)' \equiv 0$,

(ii) $\|f\| \leq \|Tf\| \leq (1 + \varepsilon)\|f\|$,

(iii) $k < (4 - \sqrt{10})/6$ and $\varepsilon < 6k^2 - 8k + 1$,

then $X$ and $Y$ are homeomorphic.

In this paper, we consider this problem from another viewpoint. To the end, we recall a closed $*$-derivation.

Let $K$ be a compact Hausdorff space and $C(K)$ denotes the space of all complex valued continuous functions on $K$ with the supremum norm $\| \cdot \|_\infty$. A closed $*$-derivation $\delta$ in $C(K)$ is a linear mapping in $C(K)$ satisfying the following conditions:

1. The domain $\mathcal{D}(\delta)$ of $\delta$ is a norm dense subalgebra of $C(K)$.
2. $\delta(fg) = \delta(f)g + f\delta(g)$ $(f, g \in \mathcal{D}(\delta))$.
3. If $f_n \in \mathcal{D}(\delta)$, $f_n \rightarrow f$, and $\delta(f_n) \rightarrow g$ implies $f \in \mathcal{D}(\delta)$ and $\delta(f) = g$ (i.e., $\delta$ is closed as a linear operator).
4. $f \in \mathcal{D}(\delta)$ implies $f^* \in \mathcal{D}(\delta)$ and $\delta(f^*) = \delta(f)^*$, where $f^*$ means the complex conjugate of $f$.

The differentiation $d/dt$ on the space $C^{1}([0, 1])$ of continuously differentiable functions on $[0, 1]$ is a typical example of closed $*$-derivations. For any closed $*$-derivation $\delta$ in $C(K)$, we may regard the domain $\mathcal{D}(\delta)$ of $\delta$ as a generalization of the Banach space $C^{1}([0, 1])$. Moreover, if $\mathcal{D}(\delta) = C(K)$, $\delta$ is bounded and hence $\delta \equiv 0$.

Properties of the domains of closed $*$-derivations have been studied by many authors.

We summarize useful properties of closed $*$-derivations which is used later frequently without references.

**Property 4** [7]. For $f(= f^*) \in \mathcal{D}(\delta)$ and $h \in C^{1}([-\|f\|_\infty, \|f\|_\infty])$, $h(f)(= h \circ f) \in \mathcal{D}(\delta)$ and $\delta(h(f)) = h'(f)\delta(f)$, where $h'$ means the derivative of $h$.

**Property 5** [7]. If $f \in \mathcal{D}(\delta)$ is a constant in a neighborhood of $x \in K$, then $\delta(f)(x) = 0$.

**Property 6** [7]. Let $J_1$ and $J_2$ be disjoint closed subsets of $K$. Then there is a function $f \in \mathcal{D}(\delta)$ such that

$$f = 0 \quad \text{on} \quad J_1, \quad f = 1 \quad \text{on} \quad J_2, \quad (0 \leq f \leq 1).$$  

(2)

Now, for any fixed point $x \in K$, we define a linear functional $\eta_x \circ \delta$ on $\mathcal{D}(\delta)$ by

$$\eta_x \circ \delta(f) := \delta(f)(x) \quad (f \in \mathcal{D}(\delta)).$$  

(3)

Let $K(\delta)$ be the set of $x \in K$ such that $\eta_x \circ \delta \neq 0$, i.e.,

$$K(\delta) = \{x \in K : \eta_x \circ \delta \neq 0\} = \{x \in K : \exists f \in \mathcal{D}(\delta) \text{ such that } \delta(f)(x) \neq 0\}.$$  

(4)

Then $K(\delta)$ is an open subset of $K$. 

Throughout this paper, the norm \(|||\) in \(\mathcal{D}(\delta)\) is given by
\[
||f|| := ||f||_\infty + ||\delta(f)||_\infty \quad (f \in \mathcal{D}(\delta)).
\] (5)

Then we note that for \(x_0 \in K(\delta)\), the norm of a linear functional \(\eta_{x_0} \circ \delta\) is 1 (see [6]).

In [6], we obtained the following result.

**Theorem 7.** Let \(K_i\) be a compact Hausdorff space and let \(\delta_i\) be a closed \(*\)-derivation in \(C(K_i)\) \((i = 1, 2)\). Let \(T\) be a surjective linear isometry between \(\mathcal{D}(\delta_1)\) and \(\mathcal{D}(\delta_2)\). Then, there exist a homeomorphism \(\tau\) from \(K_2\) to \(K_1\), \(w_1 \in \ker(\delta_2)\) and a continuous function \(w_2\) on \(K_2(\delta_2)\) such that \(\tau(K_2(\delta_2)) = K_1(\delta_1)\), \(|w_1(y)| = 1\) for all \(y \in K_2\), \(|w_2(y)| = 1\) for all \(y \in K_2(\delta_2)\), \(Tf)(y) = w_1(y)f(\tau(y))\) for \(f \in \mathcal{D}(\delta_1)\), \(y \in K_2\), \(\delta_2(Tf)(y) = w_2(y)\delta_1(f)(\tau(y))\) for \(f \in \mathcal{D}(\delta_1)\), \(y \in K_2(\delta_2)\).

In this paper, we consider Jarosz’s problem in the same context as this theorem.

We use the following notation, for a Banach space \(B, B^*\) denotes the conjugate space of \(B\). \(B_i\) and \(B_i^*\) denote the closed unit balls of \(B\) and \(B^*\), respectively. \(T\) denotes the unit circle \(\{z \in \mathbb{C} : |z| = 1\}\) in the complex plane.

We shall prove the following theorem.

**Theorem 8.** Let \(K_1\) be a compact Hausdorff space satisfying the first countable axiom, and let \(\delta_i\) be a closed \(*\)-derivation in \(C(K_i)\) \((i = 1, 2)\). If there exist a linear isomorphism \(T\) of \(\mathcal{D}(\delta_1)\) onto \(\mathcal{D}(\delta_2)\) with \(||T||||T^{-1}|| < 2\) and \(T, T^{-1}\) are bounded under the uniform norm, then \(K_1(\delta_1)\) and \(K_2(\delta_2)\) are homeomorphic. Moreover, if the range \(\mathcal{R}(\delta_i)\) contains 1 \((i = 1, 2)\), then \(K_1\) and \(K_2\) are homeomorphic.

The proof of this theorem is done along the line in [3].

Let \(K\) be a compact Hausdorff space satisfying the first countable axiom and let \(\delta\) be a closed \(*\)-derivation in \(C(K)\).

The following two lemmas will be used in the rest of the paper.

**Lemma 9.** For \(x_0 \in K(\delta)\), an open neighborhood \(U\) of \(x_0\) and \(\varepsilon\) \((0 < \varepsilon < 1)\), there exists a function \(f \in \mathcal{D}(\delta)\) such that
\[
||f|| \leq 1, \quad ||f||_\infty \leq \varepsilon, \quad f(x_0) = 0, \quad f = \delta(f) = 0 \quad \text{on} \ K \setminus U, \quad 1 > |\delta(f)(x_0)| > 1 - \varepsilon. \tag{7}
\]

**Proof.** We take an open neighborhood \(V\) of \(x_0\) such that \(V \subset U\) and take a function \(g \in \mathcal{D}(\delta)\) such that
\[
0 \leq g \leq 1, \quad g(x_0) = 1, \quad g = 0 \quad \text{on} \ K \setminus V. \tag{8}
\]

Then, \(g = \delta(g) = 0\) on \(K \setminus U\). Since \(x_0 \in K(\delta)\), there is a function \(g_\varepsilon(= g_\varepsilon^*) \in \mathcal{D}(\delta)\) such that
\[
||g_\varepsilon|| < 1, \quad 1 - \varepsilon = ||\eta_{x_0} \circ \delta|| - \varepsilon < |\delta(g_\varepsilon)(x_0)|. \tag{9}
\]
Then $f := h(g_\varepsilon)g \in \mathcal{D}(\delta)$ has all required properties in Lemma 9.

**Lemma 10.** For $x_0 \in K(\delta)$ and $\varepsilon \ (0 < \varepsilon < 1)$, there exists a sequence $\{f_n\} \subset \mathcal{D}(\delta)$ such that

$$
\|f_n\| \leq 1, \quad \|f_n\|_\infty \leq \frac{1}{n}, \quad f_n(x_0) = 0, \quad \lim_{n \to \infty} \delta(f_n)(x) = 0 \quad (x \neq x_0), \quad 1 > |\delta(f_n)(x_0)| > 1 - \varepsilon,
$$

and $d_{x_0} := \delta(f_n)(x_0)$ is independent of $n$.

**Proof.** Since $K$ satisfies the first countable axiom, there is a family $\{U_n\}$ of open neighborhood of $x_0$ such that $U_{i+1} \subset U_i$ and $\bigcap \overline{U_n} = \{x_0\}$. Then there exists a family $\{V_n\}$ of open neighborhood of $x_0$ such that $\overline{V}_n \subset U_n$, and there is $g_n \in \mathcal{D}(\delta)$ such that

$$
g_n(x_0) = 1, \quad 0 \leq g_n \leq 1, \quad g_n = 0 \text{ on } K \setminus V_n.
$$

Then $g_n = \delta(g_n) = 0$ on $K \setminus U_n$. Since $x_0$ is in $K(\delta)$, there is a function $g_\varepsilon (= g_\varepsilon^*) \in \mathcal{D}(\delta)$ such that

$$
\|g_\varepsilon\| < 1, \quad 1 - \varepsilon = \|g_{x_0} \circ \delta\| - \varepsilon < |\delta(g_\varepsilon)(x_0)|.
$$

For each $c_n := \min\{(1 - \|\delta(g_\varepsilon)\|_\infty) / (1 + \|\delta(g_n)\|_\infty), 1/n\}$, there is a function $h_n \in C^1([-\|g_\varepsilon\|_\infty, \|g_\varepsilon\|_\infty])$ such that

$$
\|h_n\|_\infty \leq c_n, \quad h_n(g_\varepsilon(x_0)) = 0, \quad h_n'(g_\varepsilon(x_0)) = 1, \quad \|h_n'\|_\infty = 1.
$$

Then every $f_n := h_n(g_\varepsilon)g_n \in \mathcal{D}(\delta)$ has the properties required in Lemma 10.

Let $W$ be the compact Hausdorff space $W = K \times K \times T$ with the product topology. For $f \in \mathcal{D}(\delta)$, we define $\tilde{f} \in C(W)$ by

$$
\tilde{f}(x,x',z) := zf(x) + \delta(f)(x'),
$$

for $(x,x',z) \in W$. Then we have $\|\tilde{f}\|_\infty = \|f\|$.

**Proof of Theorem 7.** Let $W_i := K_i \times K_i \times T$ and $S_i = \{\tilde{f} \in C(W_i); f \in \mathcal{D}(\delta_i)\}$ ($i = 1, 2$).

Define a linear isomorphism $\tilde{T}$ of $S_1$ onto $S_2$ by

$$
\tilde{T}(\tilde{f}) := \tilde{T}(f) \quad (\tilde{f} \in S_1).
$$

Then $\tilde{T}$ is well defined since $f \to \tilde{f}$ is a linear isomorphism.

We may assume that $\|T^{-1}\| = 1$ and $1 < \|T\| < 2$. Then we have $\|\tilde{T}^{-1}\| = \|T^{-1}\| = 1$ and $\|\tilde{T}\| = \|T\| < 2$. For $(y_0, y_0', z_0) \in W_2$, let $\Phi$ be a norm-preserving extension of $\tilde{T}^*L_{(y_0, y_0', z_0)}$ to $C(W_1)$, where $L_{(y_0, y_0', z_0)}$ denotes the linear functional defined by
Then for Lemma 10. Then for
for  \( \Phi \)

The following lemma shows that for \( \Phi \), \( f \in \mathcal{D}(\delta_1) \).

In the following, we identify \( \Phi \) and \( \mu^{\gamma_0, \gamma_0', z_0} \),

\( \mu^{x_0, x_0', z_0} \), where \( (x_0, x_0', z_0) \in W_1 \), is also defined in a similar way. Then we have

\[ \| \mu^{x_0, x_0', z_0} \| \leq 1. \]

The following lemma shows that for \( x_0 \in K_1(\delta_1) \), \( \mu^{\gamma, \gamma', z}(K_1 \times \{x_0\} \times T) \), where \( (\gamma, \gamma', z) \in W_2 \) depends on \( \gamma' \) only, that is, \( \mu^{\gamma, \gamma', z}(K_1 \times \{x_0\} \times T) \) is independent of \( \gamma \), \( z \), and any choice of norm-preserving extension of \( \tilde{T} \). L(\gamma, \gamma', z).

**Lemma 11.** (1) For \( x_0 \in K_1(\delta_1) \) and \( \epsilon (0 < \epsilon < 1) \), let \( \{f_n\} \subset \mathcal{D}(\delta_1) \) be a sequence in Lemma 10. Then for \( (\gamma, \gamma', z) \in W_2 \),

\[ \mu^{\gamma, \gamma', z}(K_1 \times \{x_0\} \times T) = \left( \frac{1}{d_{x_0}} \right) \lim_{n \to \infty} \tilde{T}(f_n)(\gamma, \gamma', z) \]

\[ = \left( \frac{1}{d_{x_0}} \right) \lim_{n \to \infty} \delta_2(T(f_n))(\gamma'). \]

(2) For \( y_0 \in K_2(\delta_2) \) and \( \epsilon (0 < \epsilon < 1) \), let \( \{g_n\} \subset \mathcal{D}(\delta_2) \) be a sequence in Lemma 10. Then for \( (x, x', z) \in W_1 \),

\[ \mu^{\gamma, \gamma', z}(K_2 \times \{y_0\} \times T) = \left( \frac{1}{d_{y_0}} \right) \lim_{n \to \infty} \tilde{T}^{-1}(g_n)(x, x', z) \]

\[ = \left( \frac{1}{d_{y_0}} \right) \lim_{n \to \infty} \delta_1(T^{-1}(g_n))(x'). \]

**Proof.** (1) Let \( \mu^{\gamma, \gamma', z} \) be a norm-preserving extension of \( \tilde{T} \).

\[ \lim_{n \to \infty} \tilde{T}(f_n)(\gamma, \gamma', z) = \lim_{n \to \infty} \int_{W_1} f_n \, d\mu^{\gamma, \gamma', z} = \int_{W_1} \lim_{n \to \infty} f_n \, d\mu^{\gamma, \gamma', z} \]

\[ = \int_{K_1 \times \{x_0\} \times T} d_{x_0} \, d\mu^{\gamma, \gamma', z} = d_{x_0} \mu^{\gamma, \gamma', z}(K_1 \times \{x_0\} \times T). \]

From the uniform boundedness of \( T \),

\[ \lim_{n \to \infty} \tilde{T}(f_n)(\gamma, \gamma', z) = \lim_{n \to \infty} (T(f_n)(\gamma) + \delta_2(T(f_n))(\gamma')) = \lim_{n \to \infty} \delta_2(T(f_n))(\gamma'). \]
Thus, we have

\[d_{x_0} \mu^{y',z}(K_1 \times \{x_0\} \times T) = \lim_{n \to \infty} \delta_2(T f_n) (y')\] (23)

which implies that for \(x_0 \in K_1(\delta_1), \mu^{y',z}(K_1 \times \{x_0\} \times T)\) depends on \(y' \in K_2\) only.

The statement (2) is also shown by the same argument as above.

Now, let \(M_1\) be any real number with \((1 <) \|T\| < 2M_1 < 2\). Let \(\tilde{K}_2 := \{y \in K_2 : \exists x \in K_1 \text{ such that } |\mu^{y',z}(K_1 \times \{x\} \times T)| > M_1\} \text{ for every } z \in T\) and every norm-preserving extension \(\mu^{y',z}\) of \(T^* L_{(y,y',z)}\). Since \(\|\mu^{y',z}\| = \|T^* L_{(y,y',z)}\| \leq \|T\| < 2M_1\), for \(y \in \tilde{K}_2\), there can be at most one \(x \in K_1\) with the property in the definition of \(\tilde{K}_2\). Thus the map \(\rho_1\) of \(\tilde{K}_2\) to \(K_1\) is well defined by \(\rho_1(y) := x\) if \(x\) is related to \(y\) as above.

Next, we set \(M_2 := 1/(2M_1)\). Let \(\tilde{K}_1 := \{x \in K_1 : \exists y \in K_2 \text{ such that } |\mu^{x,y,z}(K_2 \times \{y\} \times T)| > M_2\} \text{ for every } z \in T\) and for every norm-preserving extension \(\mu^{x,y,z}\) of \((T^{-1})^* L_{(x,x,z)}\). Since \(\|\mu^{x,y,z}\| = \|T^{-1})^* L_{(x,x,z)}\| \leq \|T^{-1}\| < 1\), for \(x \in \tilde{K}_1\), there can be at most one \(y \in K_2\) with the property in the definition of \(\tilde{K}_1\). Thus, the map \(\rho_2\) of \(\tilde{K}_1\) to \(K_2\) is well defined by \(\rho_2(x) := y\) if \(y\) is related to \(x\) as above.

The following lemma shows that \(\tilde{K}_1\) contains sufficiently many elements (hence, is nonempty).

**Lemma 12.** (1) For \(x_0 \in K_1(\delta_1)\), there exists \(y_0 \in \tilde{K}_2 \cap K_2(\delta_2)\) such that \(\rho_1(y_0) = x_0\).

(2) For \(y_0 \in K_2(\delta_2)\), there exists \(x_0 \in \tilde{K}_1 \cap K_1(\delta_1)\) such that \(\rho_2(x_0) = y_0\).

**Proof.** (1) For \(x_0 \in K_1(\delta_1)\) and \(0 < \varepsilon < 1-M_1\), there exists a family \(\{f_n\} \subset \mathcal{D}(\delta_1)\) in Lemma 10 such that

\[
\|f_n\| \leq 1, \quad \|f_n\|_\infty \leq \frac{1}{n}, \quad f_n(x_0) = 0, \quad \lim_{n \to \infty} \delta_1(f_n)(x) = 0 \quad (\forall x \neq x_0), \quad 1 - \varepsilon < \|d_{x_0}\| < 1,
\]

where \(d_{x_0} = \delta_1(f_n)(x_0)\). If \(\lim_{n \to \infty} |\tilde{T}(\tilde{f}_n)(y,y',z)| \leq M_1\) for every \((y,y',z) \in W_2\), then

\[
1 - \varepsilon < \|d_{x_0}\| = \lim_{n \to \infty} |f_n(x_0) + \delta_1(f_n)(x_0)| = \lim_{n \to \infty} |\tilde{f}_n(x_0,x_0,1)|
\]

\[
= \lim_{n \to \infty} \left(\tilde{T}^{-1})^* L_{(x_0,x_0,1)}(\tilde{T}(\tilde{f}_n))\right)
\]

\[
= \lim_{n \to \infty} \left|\int_{W_2} \tilde{T}(\tilde{f}_n)(y,y',z) d\mu^{x_0,x_0,1}\right|
\]

\[
\leq \int_{W_2} \lim_{n \to \infty} |\tilde{T}(\tilde{f}_n)(y,y',z)| \cdot |d\mu^{x_0,x_0,1}|
\]

\[
\leq M_1 |\mu^{x_0,x_0,1}| \leq M_1.
\]

This contradicts with \(1 - \varepsilon > M_1\).

Hence there exists \((y_0,y'_0,z_0) \in W_2\) such that

\[
\lim_{n \to \infty} |\tilde{T}(\tilde{f}_n)(y_0,y'_0,z_0)| > M_1.
\] (26)
Then, from Lemma 11 we have for arbitrary \( z \in \mathbb{T} \) and any norm-preserving extension \( \mu^{y_0,y_0,z} \) of \( \tilde{T}^*L_{(y_0,y_0,z)} \),

\[
M_1 < \lim_{n \to \infty} |\tilde{T}(\tilde{f}_n)(y_0,y_0,z)| = \lim_{n \to \infty} |\delta_2(Tf_n)(y_0)| = \lim_{n \to \infty} |\tilde{T}(\tilde{f}_n)(y_0,y_0,z)| = |d_{x_0}\mu^{y_0,y_0,z}(K_1 \times \{x_0\} \times \mathbb{T})| < |\mu^{y_0,y_0,z}(K_1 \times \{x_0\} \times \mathbb{T})|.
\]

(27)

Thus, \( y_0' \in \tilde{K}_2 \cap K_2(\delta_2) \) and \( \rho_1(y_0') = x_0 \).

Now, we state another important lemma which holds without the first countability axiom.

**Lemma 13.** If \( x_0 \in \tilde{K}_1 \) and \( \rho_2(x_0) \in K_2(\delta_2) \), then \( x_0 \in K_1(\delta_1) \).

**Proof.** Let \( \mu^{x_0,x_0,1} \) be a norm-preserving extension of \( \tilde{T}^{-1}L_{(x_0,x_0,1)} \). Since \( \mu^{x_0,x_0,1} \) is regular, since for all \( \varepsilon \) such that \( 0 < \varepsilon < M_2/\mu^{x_0,x_0,1} \), there is an open neighborhood \( U_\varepsilon \) of \( \rho_1(x_0) \) such that

\[
|\mu^{x_0,x_0,1}|(K_2 \times (U_\varepsilon \setminus \{\rho_2(x_0)\}) \times \mathbb{T}) < \varepsilon.
\]

(28)

For \( \varepsilon, U_\varepsilon \) and \( \rho_2(x_0) \), we take a function \( f \in \mathcal{D}(\delta_2) \) in Lemma 9, then

\[
\|f\| \leq 1, \quad \|f\|_\infty \leq \varepsilon, \quad f(\rho_2(x_0)) = 0,
\]

\[
f = \delta_2(f) = 0 \quad \text{on} \quad K_2 \setminus U_\varepsilon, \quad 1 > |\delta_2(f)(\rho_2(x_0))| > 1 - \varepsilon.
\]

(29)

Since

\[
\left| \int_{K_2 \times \{\rho_2(x_0)\} \times \mathbb{T}} z f(y) d\mu^{x_0,x_0,1} \right| \leq \|f\|_\infty \|\mu^{x_0,x_0,1}\| \leq \varepsilon,
\]

\[
\left| \int_{K_2 \times \{\rho_2(x_0)\} \times \mathbb{T}} \delta_2(f)(\rho_2(x_0)) d\mu^{x_0,x_0,1} \right| = |\delta_2(f)(\rho_2(x_0))|\|\mu^{x_0,x_0,1}\|(|K_2 \times \{\rho_2(x_0)\} \times \mathbb{T})| > (1 - \varepsilon)M_2,
\]

(30)

we have

\[
\left| \int_{K_2 \times \{\rho_2(x_0)\} \times \mathbb{T}} \tilde{f} d\mu^{x_0,x_0,1} \right| \geq \left| \int_{K_2 \times \{\rho_2(x_0)\} \times \mathbb{T}} \delta_2(f)(\rho_2(x_0)) d\mu^{x_0,x_0,1} \right| - \left| \int_{K_2 \times \{\rho_2(x_0)\} \times \mathbb{T}} z f(y) d\mu^{x_0,x_0,1} \right| > (1 - \varepsilon)M_2 - \varepsilon > 0.
\]

(31)

From this and

\[
\left| \int_{K_2 \times (U_\varepsilon \setminus \{\rho_2(x_0)\}) \times \mathbb{T}} \tilde{f} d\mu^{x_0,x_0,1} \right| \leq \|\tilde{f}\|_\infty \|\mu^{x_0,x_0,1}\|(|K_2 \times (U_\varepsilon \setminus \{\rho_2(x_0)\}) \times \mathbb{T})| \leq \varepsilon,
\]

(32)
we have

\[ \left| \int_{K_2 \times U_2 \times T} \tilde{f} \, d\mu^{x_0,z_0} \right| \geq \left| \int_{K_2 \times \{\rho_2(x_0)\} \times T} \tilde{f} \, d\mu^{x_0,z_0} \right| - \left| \int_{K_2 \times \{U_2\} \times T} \tilde{f} \, d\mu^{x_0,z_0} \right| \]

(33)

\[ > (1 - \varepsilon)M_2 - 2\varepsilon > 0. \]

Since

\[ \left| \int_{K_2 \times (K_2 \setminus U_2) \times T} \tilde{f} \, d\mu^{x_0,z_0} \right| = \left| \int_{K_2 \times (K_2 \setminus U_2) \times T} z f(y) \, d\mu^{x_0,z_0} \right| \]

\[ \leq \|f\|_{\infty} \|\mu^{x_0,z_0}\| \leq \varepsilon, \]

we get

\[ \left| (\tilde{T}^{-1} \tilde{f})(x_0,x_0,1) \right| = \left| (\tilde{T}^{-1})^* L_{(x_0,x_0,1)}(\tilde{f}) \right| = \left| \int_{W_2} \tilde{f} \, d\mu^{x_0,z_0} \right| \]

\[ \geq \left| \int_{K_2 \times U_2 \times T} \tilde{f} \, d\mu^{x_0,z_0} \right| - \left| \int_{K_2 \times (K_2 \setminus U_2) \times T} \tilde{f} \, d\mu^{x_0,z_0} \right| \]

(35)

\[ \geq (1 - \varepsilon)M_2 - 3\varepsilon > 0. \]

Thus

\[ \left| \delta_1(T^{-1}(f))(x_0) \right| = \left| \tilde{T}^{-1}(\tilde{f})(x_0,x_0,1) - T^{-1}(f)(x_0) \right| \]

\[ \geq \left| \tilde{T}^{-1}(\tilde{f})(x_0,x_0,1) \right| - \left| T^{-1}(f)(x_0) \right| \]

(36)

\[ \geq (1 - \varepsilon)M_2 - \varepsilon\|T^{-1}\|_{\infty} > 0, \]

that is, \( x_0 \in K_1(\delta_1) \). This completes the proof. \( \square \)

**Lemma 14.** If \( y_0 \in \tilde{K}_2 \cap K_2(\delta_2) \), then \( \rho_1(y_0) \in \tilde{K}_1 \cap K_1(\delta_1) \) and \( \rho_2(\rho_1(y_0)) = y_0 \).

**Proof.** Let \( \rho_1(y_0) = x_0 \) (\( y_0 \in \tilde{K}_2 \cap K_2(\delta_2) \)). If \( x_0 \in \tilde{K}_1 \) and \( \rho_2(x_0) = y_0 \), then \( x_0 \in K_1(\delta_1) \) from Lemma 13. Hence, suppose that either \( x_0 \) is not in \( \tilde{K}_1 \) or \( x_0 \in \tilde{K}_1 \) and \( \rho_2(x_0) \neq y_0 \). Then there exists \( z_0 \in T \) such that \( |\mu^{x_0,z_0}(K_2 \times \{y_0\} \times T)| \leq M_2 \).

Let \( P := \sup\{|\mu^{x',x,z}(K_2 \times \{y_0\} \times T)|; (x,x',z) \in W_1\}(\leq 1) \). Since \( y_0 \in K_2(\delta_2) \), we have \( P = \sup\{|\mu^{x',x,z}(K_2 \times \{y_0\} \times T)|; (x,x',z) \in W_1\} \) by Lemma 11. Since \( P > M_2 \) by Lemma 12 and \( 0 < \|T\| - M_1 < M_1 \), there exists \( (x_1,x_1,z_1) \in W_1 \) such that

\[ |\mu^{x_1,x_1,z_1}(K_2 \times \{y_0\} \times T)| > \max \{M_2, (\|T\| - M_1)P/M_1\}. \]

(37)

Then, for arbitrary \( z \in T \) and any norm-preserving extension \( \mu^{x_1,x_1,z} \),

\[ |\mu^{x_1,x_1,z}(K_2 \times \{y_0\} \times T)| > M_2, \]

(38)

by Lemma 11. Thus, \( x_1 \in \tilde{K}_1 \), \( \rho_2(x_1) = y_0 \), and \( x_1 \neq x_0 \). Therefore, \( x_1 \in K_1(\delta_1) \) by Lemma 13. Since \( x_1 \neq x_0 \), there exist \( y_1(\neq y_0) \in \tilde{K}_2 \cap K_2(\delta_2) \) such that \( \rho_1(y_1) = x_1 \).
by Lemma 12. For \( y_0 \in K_2(\delta_2) \) and \( \varepsilon (0 < \varepsilon < 1) \), there exists a family \( \{ g_n \} \subset \mathbb{D}(\delta_2) \) in Lemma 10. Then, since \( y_1 \neq y_0 \),

\[
0 = \lim_{n \to \infty} (z_1 g_n(y_1) + \delta_2(\varrho_n)(y_1)) = \lim_{n \to \infty} \varrho_n(y_1, y_1, z_1)
\]

\[
= \lim_{n \to \infty} \bar{T}^* L(y_1, y_1, z_1)(\bar{T}^{-1}(\varrho_n)) = \lim_{n \to \infty} \int_{W_1} \bar{T}^{-1}(\varrho_n) d\mu^{y_1, y_1, z_1}
\]

\[
= \lim_{n \to \infty} \int_{K_1 \times [x_1] \times T} \bar{T}^{-1}(\varrho_n) d\mu^{y_1, y_1, z_1} + \lim_{n \to \infty} \int_{K_1 \times (K_1 \setminus \{ x_1 \}) \times T} \bar{T}^{-1}(\varrho_n) d\mu^{y_1, y_1, z_1}.
\]

(39)

Now, by Lemma 11,

\[
\left| \lim_{n \to \infty} \int_{K_1 \times [x_1] \times T} \bar{T}^{-1}(\varrho_n) d\mu^{y_1, y_1, z_1} \right|
\]

\[
= \left| \int_{K_1 \times [x_1] \times T} \lim_{n \to \infty} \bar{T}^{-1}(\varrho_n) d\mu^{y_1, y_1, z_1} \right|
\]

\[
= \int_{K_1 \times [x_1] \times T} d_{y_0} \mu^{x_1, x_1}(K_2 \times \{ y_0 \} \times T) d\mu^{y_1, y_1, z_1}
\]

\[
= \int_{K_1 \times [x_1] \times T} d_{y_0} \mu^{x_1, x_1}(K_2 \times \{ y_0 \} \times T) d\mu^{y_1, y_1, z_1}
\]

(40)

(41)

On the other hand,

\[
\left| \lim_{n \to \infty} \int_{K_1 \times (K_1 \setminus \{ x_1 \}) \times T} \bar{T}^{-1}(\varrho_n) d\mu^{y_1, y_1, z_1} \right|
\]

\[
= \left| \int_{K_1 \times (K_1 \setminus \{ x_1 \}) \times T} \lim_{n \to \infty} \bar{T}^{-1}(\varrho_n) d\mu^{y_1, y_1, z_1} \right|
\]

\[
= \int_{K_1 \times (K_1 \setminus \{ x_1 \}) \times T} d_{y_0} \mu^{x_1, x_1}(K_2 \times \{ y_0 \} \times T) d\mu^{y_1, y_1, z_1}
\]

\[
\leq |d_{y_0}| P(\mu^{y_1, y_1, z_1} | (K_1 \times (K_1 \setminus \{ x_1 \}) \times T)
\]

\[
= |d_{y_0}| P(\mu^{y_1, y_1, z_1} | (K_1 \times (K_1 \setminus \{ x_1 \}) \times T)
\]

\[
\leq |d_{y_0}| P(\| T \| - |\mu^{y_1, y_1, z_1} | (K_1 \times (K_1 \setminus \{ x_1 \}) \times T)) < |d_{y_0}| P(\| T \| - M_1).
\]

This contradicts to

\[
0 = \lim_{n \to \infty} \int_{K_1 \times [x_1] \times T} \bar{T}^{-1}(\varrho_n) d\mu^{y_1, y_1, z_1} + \lim_{n \to \infty} \int_{K_1 \times (K_1 \setminus \{ x_1 \}) \times T} \bar{T}^{-1}(\varrho_n) d\mu^{y_1, y_1, z_1}.
\]

(42)

Thus \( x_0 \in \mathring{K}_1 \) and \( y_0 = \rho_2(x_0) = \rho_2(\rho_1(y_0)) \).
**Lemma 15.** \( \rho_i \) is continuous on \( K_i(\delta_i) \) \((i = 1, 2)\).

**Proof.** We show that \( \rho_1 \) is continuous. Suppose that \( \rho_1 \) is discontinuous at \( y_0 \in K_2(\delta_2) \). Then there exists a sequence \( \{y_n\} \subset K_2(\delta_2) \) such that \( y_n \to y_0 \in K_2(\delta_2) \), but \( x_n := \rho_1(y_n) \) is not converge to \( \rho_1(y_0) = x_0 \). There exists an open neighborhood \( V_1(\subseteq K_1(\delta_1)) \) of \( x_0 \) such that for every \( n_0 \) there is \( n(n \geq n_0) \) with \( x_n \) outside \( V_1 \). Since \( \mu_{y_0,y_0} \) is regular, for \( \epsilon > 0 \) \((0 < \epsilon < (2M_1 - \|T\|)/(\|T\| + 2M_1 + 10)\)) there exists an open neighborhood \( U_1(\subseteq V_1) \) of \( x_0 \) such that

\[
\|\mu_{y_0,y_0} \| (K_1 \times (U_1 \setminus \{x_0\}) \times T) < \epsilon, \quad U_1 \subset V_1.
\]

(43)

For \( x_0, U_1, \) and \( \epsilon, \) by Lemma 9, there exists a function \( f \in \mathcal{D}(\delta_1) \) such that

\[
\|f\| \leq 1, \quad \|f\| \leq \epsilon, \quad f(x_0) = 0,
\]

(44)

\[
1 > |\delta_1(f)(x_0)| > 1 - \epsilon, \quad f = \delta_1(f) = 0 \quad \text{on } K_1 \setminus U_1.
\]

Since

\[
\left| \int_{K_1 \times \{x_0\} \times T} z f(x) d\mu_{y_0,y_0} \right| \leq \|f\| \|\mu_{y_0,y_0}\| \leq 2\epsilon,
\]

\[
\left| \int_{K_1 \times \{x_0\} \times T} \delta_1(f)(x_0) d\mu_{y_0,y_0} \right| = |\delta_1(f)(x_0)| \|\mu_{y_0,y_0}\| (K_1 \times \{x_0\} \times T) \geq (1 - \epsilon)M_1,
\]

we have

\[
\left| \int_{K_1 \times \{x_0\} \times T} \tilde{f} d\mu_{y_0,y_0} \right| > (1 - \epsilon)M_1 - 2\epsilon > \epsilon.
\]

(46)

From (46) and

\[
\left| \int_{K_1 \times (U_1 \setminus \{x_0\}) \times T} \tilde{f} d\mu_{y_0,y_0} \right| \leq \|\tilde{f}\| \|\mu_{y_0,y_0}\| (K_1 \times (U_1 \setminus \{x_0\}) \times T) \leq \epsilon,
\]

(47)

we have

\[
\left| \int_{K_1 \times U_1 \times T} \tilde{f} d\mu_{y_0,y_0} \right| \geq \left| \int_{K_1 \times \{x_0\} \times T} \tilde{f} d\mu_{y_0,y_0} \right| - \left| \int_{K_1 \times (U_1 \setminus \{x_0\}) \times T} \tilde{f} d\mu_{y_0,y_0} \right| \geq (1 - \epsilon)M_1 - 3\epsilon > 2\epsilon
\]

(48)

\[
\left| \int_{K_1 \times (K_1 \setminus U_1) \times T} \tilde{f} d\mu_{y_0,y_0} \right| = \left| \int_{K_1 \times (K_1 \setminus U_1) \times T} z f(x) d\mu_{y_0,y_0} \right| \leq \|f\| \|\mu_{y_0,y_0}\| \leq 2\epsilon.
\]
Thus

\[ |\tilde{T}(\tilde{f})(y_0, y_0, 1)| = |\tilde{T}^* L_{(y_0, y_0, 1)}(\tilde{f})| = \left| \int_{W_1} \tilde{f} d\mu^{y_0, y_0, 1} \right| \geq \left| \int_{K_1 \times U_1 \times T} \tilde{f} d\mu^{y_0, y_0, 1} \right| - \left| \int_{K_1 \times (K_1 \setminus U_1) \times T} \tilde{f} d\mu^{y_0, y_0, 1} \right| \]

\[ > (1 - \varepsilon)M_1 - 5\varepsilon > 0. \]

Now, since \( y_n - y_0 \) in \( K_2 \), then \((y_n, y_n, 1) \to (y_0, y_0, 1) \) in \( W_2 \). There exists \( n_0 \) such that \( \forall n > n_0 \) implies \( |\tilde{T}(\tilde{f})(y_n, y_n, 1)| > (1 - \varepsilon)M_1 - 5\varepsilon \). Fix \( n_1 \geq n_0 \) such that \( x_{n_1} = \rho_1(y_{n_1}) \) lies outside \( V_1 \). Since \( \mu^{y_n, y_n, 1} \) is regular, there exists an open neighborhood \( U_2(\subset K_1) \) of \( x_{n_1} \) such that

\[ |\mu^{y_{n_1}, y_{n_1}, 1}|(K_1 \times (U_2 \setminus \{x_{n_1}\}) \times T) < \varepsilon, \quad \overline{U_1} \cap U_2 = \phi. \]

For \( x_{n_1}, U_2, \) and \( \varepsilon \), we take \( g(\in \Omega(\delta_1)) \) in Lemma 9 such that

\[ \|g\| \leq 1, \quad \|g\|_\infty \leq \varepsilon, \quad g(x_{n_1}) = 0, \]

\[ 1 > |\delta_1(g)(x_{n_1})| > 1 - \varepsilon, \quad g = \delta_1(g) = 0 \quad \text{on } K_1 \setminus U_2. \]

By the same way as above, we have

\[ \left| \int_{K_1 \times U_2 \times T} \tilde{g} d\mu^{y_{n_1}, y_{n_1}, 1} \right| > (1 - \varepsilon)M_1 - 3\varepsilon > 0, \]

\[ \left| \int_{K_1 \times (K_1 \setminus U_2) \times T} \tilde{g} d\mu^{y_{n_1}, y_{n_1}, 1} \right| = \left| \int_{K_1 \times (K_1 \setminus U_2) \times T} z g(x) d\mu^{y_{n_1}, y_{n_1}, 1} \right| \]

\[ \leq \|g\|_\infty \|\mu^{y_{n_1}, y_{n_1}, 1}\| \leq 2\varepsilon. \]

Then

\[ |\tilde{T}(\tilde{g})(y_{n_1}, y_{n_1}, 1)| = |\tilde{T}^* L_{(y_{n_1}, y_{n_1}, 1)}(\tilde{g})| = \left| \int_{W_1} \tilde{g} d\mu^{y_{n_1}, y_{n_1}, 1} \right| \geq \left| \int_{K_1 \times U_2 \times T} \tilde{g} d\mu^{y_{n_1}, y_{n_1}, 1} \right| - \left| \int_{K_1 \times (K_1 \setminus U_2) \times T} \tilde{g} d\mu^{y_{n_1}, y_{n_1}, 1} \right| \]

\[ > (1 - \varepsilon)M_1 - 5\varepsilon > 0. \]

Thus, if we choose a complex number \( \lambda_0 \in T \) such that \( \tilde{T}(\tilde{f})(y_{n_1}, y_{n_1}, 1) \) and \( \lambda_0 \) \((\tilde{T}(\tilde{g}))(y_{n_1}, y_{n_1}, 1)\) have equal arguments, then

\[ \|f + \lambda_0 g\| = \max \{\|f\|_\infty, \|g\|_\infty\} + \max \{\|\delta_1(f)\|_\infty, \|\delta_1(g)\|_\infty\} \leq 1 + \varepsilon, \]

\[ \text{(54)} \]

This is a contradiction. Therefore, \( \rho_1 \) is continuous on \( K_2(\delta_2) \). A similar argument shows that \( \rho_2 \) is continuous on \( K_1(\delta_2) \).

From Lemma 15, it follows that \( K_1(\delta_1) \) and \( K_2(\delta_2) \) are homeomorphic. Thus, all proofs of Theorem are completed.
There is not a nonzero closed $\ast$-derivation in $C(D)$ ($D$ is the Cantor set). However, we can obtain similar results for $C^{(1)}(X)$ ($X$ : a compact subset of $\mathbb{R}$) by the same way as above.

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Toshiko Matsumoto: 2-19-1143 Ikego Zushi, 249-0003, Japan

Seiji Watanabe: Niigata Institute of Technology, 1719 Fujihashi, Kashiwazaki 945-1195, Japan