\textbf{T}_{\Omega^{\Omega}}\text{-SEQUENCES IN ABELIAN GROUPS}

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\textbf{Abstract.} A sequence in an abelian group is called a \( T \)-sequence if there exists a Hausdorff group topology in which the sequence converges to zero. This paper describes the fundamental system for the finest group topology in which this sequence converges to zero. A sequence is a \( T_{\Omega^{\Omega}} \)-sequence if there exist uncountably many different Hausdorff group topologies in which the sequence converges to zero. The paper develops a condition which insures that a sequence is a \( T_{\Omega^{\Omega}} \)-sequence and examples of \( T_{\Omega^{\Omega}} \)-sequences are given.

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1. Introduction. Let \( G \) be an abelian group and let \( \langle a_n \rangle_{n=1}^\infty \) be a nontrivial sequence in \( G \). If 0 is the identity element in \( G \), we can ask what is the finest group topology on \( G \) such that \( \langle a_n \rangle_{n=1}^\infty \) converges to zero? In the terminology of [2], we are placing the topology of a nonconstant sequence on the subspace \( \{a_n\}_{n=1}^\infty \cup \{0\} \subseteq G \) and finding the associated Graev topology. When this topology is Hausdorff, Zelenyuk, and Protasov [4] say that \( \langle a_n \rangle_{n=1}^\infty \) is a \( T \)-sequence. The purpose of this paper will be to extend some of the results of Zelenyuk and Protasov concerning \( T \)-sequences in specific abelian groups. We will develop a fundamental system approach to defining group topologies and use this approach to consider the cardinality of the set of Hausdorff group topologies in which a specific sequence converges to zero. This extends results found in [1].

We assume as additional hypothesis throughout this paper that \( G \) is an abelian group and that each sequence under consideration is a one-to-one function from the natural numbers \( \mathbb{N} \) into \( G \). Also the notations \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \), and \( S^1 \) will denote the integers, rationals, real, and the circle group, respectively. The subgroup of \( S^1 \) which is the set of solutions of the form \( k/p^n \), where \( k \in \mathbb{Z} \), \( p \) is prime and \( n \in \mathbb{N} \), we will denote it as \( \mathbb{Z}(p^n) \).

2. Fundamental systems generated by sequences. Since \( G \) is abelian it is possible to define various fundamental systems in a subgroup and use them as a fundamental system for the entire group. We shall use the terms of the sequence \( \langle a_n \rangle_{n=1}^\infty \) to define such a fundamental system for the subgroup generated by \( \{a_n\}_{n=1}^\infty \). Let \( T(n) = \{0\} \cup \{a_k\}_{k=n}^\infty \cup \{-a_k\}_{k=n}^\infty \), where \( -a_k \) denotes the inverse of \( a_k \) in \( G \), and let \( \zeta \) denote the collection of all increasing sequences in \( \mathbb{N} \). Then for \( C, D \in \zeta \) we define \( U(C,D) = \{g_1 + g_2 + \cdots + g_k \mid g_i \in c_i T(d_i) \text{ for } i \in \{1,2,\ldots,k\}; \ k \in \mathbb{N}\} \).

\textbf{Proposition 2.1.} \( \Sigma = \{U(C,D) \mid C,D \in \zeta\} \) is a fundamental system for \( G \).
PROOF. Suppose that \( U(C,D) \) and \( U(C',D') \) are elements of \( \mathcal{F} \). For each \( i \in \mathbb{N} \) let \( c'' = \min \{ c_1, c'_i \} \) and \( d'' = \max \{ d_i, d'_i \} \). Define \( C'' = \langle c'' \rangle_{i=1}^{\infty} \) and \( D'' = \langle d'' \rangle_{i=1}^{\infty} \). Clearly, both \( C'', D'' \in \zeta \). Since \( c_i T(n) \subseteq c_2 T(n) \) whenever \( c_1 \leq c_2 \) and \( T(n) \subseteq T(m) \) whenever \( m \leq n \), we have that \( c''_i T(d''_i) \subseteq c_i T(d_i) \cap c_i T(d'_i) \). Therefore we have \( U(C'',D'') \subseteq U(C,D) \cap U(C',D') \).

Now suppose \( x \in U(C,D) \). Then \( x = g_1 + g_2 + \cdots + g_k \) for some \( k \in \mathbb{N} \) and each \( g_i \in c_i T(d_i) \) for \( i \in \{1,2,\ldots,k\} \). If \( C' = \langle c_{k+1}, c_{k+2}, \ldots \rangle \) and \( D' = \langle d_{k+1}, d_{k+2}, \ldots \rangle \) then \( x + U(C', D') \subseteq U(C, D) \).

Let \( U(C,D) \in \mathcal{F} \). For each \( i \in \mathbb{N} \) we define

\[
c'_i = \begin{cases} \frac{c_{2i}}{2} & \text{if } c_{2i} \text{ is even}, \\ \frac{c_{2i} - 1}{2} & \text{if } c_{2i} \text{ is odd}. \end{cases}
\]

(2.1)

If \( C' = \langle c'_i \rangle \) then \( C' \in \zeta \) since \( C \in \zeta \). Also we have that \( 2c' \leq c_{2i} \) for all \( i \in \mathbb{N} \). Define \( D' = \langle d_{2i} \rangle \). Then for each \( i \in \mathbb{N} \) we have that \( 2c'_i T(d_{2i}) \subseteq c_i T(d_{2i}) \) and hence \( 2U(C', D') \subseteq U(C, D) \).

Finally, we note that since \( U(C, D)^{-1} = U(C, D) \), \( \mathcal{F} \) is a fundamental system.

PROPOSITION 2.2. The group topology generated by \( \mathcal{F} \) is the finest group topology on \( G \) for which \( \langle a_n \rangle_{n=1}^{\infty} \) converges to zero.

PROOF. Let \( \tau \) be any group topology on \( G \) for which the sequence \( \langle a_n \rangle_{n=1}^{\infty} \) converges to zero and let \( 0 \in \mathcal{T} \). We inductively define a sequence of open sets in \( \tau \), say \( V_1, V_2, \ldots, \) with \( 0 \notin V_i \) for all \( i \), \( 2V_1 \subseteq W \), and in general \( (n+1)V_n \subseteq V_{n-1} \) for \( n \geq 2 \). We also may assume that each \( V_i \) is symmetric.

For any \( k \in \mathbb{N} \) we have that \( V_1 + 2V_2 + \cdots + kV_k \subseteq W \). Since \( \langle a_n \rangle_{n=1}^{\infty} \) converges to zero in \( \tau \), we can find a tail of the sequence in \( V_i \). We choose \( d_i \in \mathbb{N} \) so that \( T(d_i) \subseteq V_i \) and \( d_i > \max \{ d_1, \ldots, d_{i-1} \} \). Then we have that \( kT(d_i) \subseteq kV_k \) and for \( D = \langle d_i \rangle \), we have that \( U(\mathbb{N}, D) \subseteq W \).

The technique used in Proposition 2.1 can be used to show that various subcollections of \( \mathcal{F} \) are also fundamental systems for \( G \). For example if \( D = \langle d_i \rangle \in \zeta \) and for \( k \in \mathbb{N}, D_k = \langle d_{ki} \rangle \), then \( \mathcal{F}' = \{ U(C, D_k) \mid C \in \zeta, \ k \in \mathbb{N} \} \) will also form a fundamental system.

TΩ-SEQUENCES. Shelah [3] constructs an example of a nonabelian group that admits only the discrete and indiscrete topologies as group topologies. Certainly, the constant identity sequence in Shelah’s group will be a T-sequence which converges in a unique Hausdorff group topology. On the other hand, the sparse sequences in \( \mathbb{Q} \) described in [1] are shown to converge to the identity in uncountably many different Hausdorff group topologies. We will call any such sequence a TΩ-sequence. As we shall see in this section, many sequences in abelian groups are actually TΩ-sequences.

Our search for TΩ-sequences will require that we focus our attention on various subcollections of the fundamental system described in Proposition 2.1. To refine our notation we define for \( D = \langle d_i \rangle \in \zeta \), \( U(\langle d_n \rangle) = \{ \sum_{i=1}^{n} g_i \mid g_i \in T(d_i) \} \) for \( i \in \{1,2,\ldots,n\} \) and \( n \in \mathbb{N} \) and \( \mathcal{F}_D = \{ U(\langle d_{kn} \rangle) \mid k \in \mathbb{N} \} \). Using techniques similar to those used in Proposition 2.1, it can be shown that \( \mathcal{F}_D \) is a fundamental system for \( G \).
We will also focus on a subcollection of $\zeta$. For each $c \in \mathbb{R}$ with $c > 2$ we define $c_n = \lfloor n^c \rfloor$, the greatest integer in $n^c$. Clearly $C = \{c_n\}_{n=1}^{\infty} \in \zeta$.

**Lemma 2.3.** If $c, d$ are real numbers with $2 < c < d$ and if $k \in \mathbb{N}$ then we can find $N_k \in \mathbb{N}$ such that for $m \geq N_k$, $c_k + m < d_m$.

**Proof.** We can find $N_k \in \mathbb{N}$ such that for $m \geq N_k$ we have that $k^c + 2 < m^{d-c}$. Hence $[(km)^c] + m < [m^d]$ and thus $c_k + m < d_m$ for all $m > N_k$.

**Definition 2.4.** Let $S \subseteq G$. For $n \in \mathbb{N}$ and $g \in G$ we say that $g$ has an $n$-factorization in $S$ if and only if there exists $\{s_1, \ldots, s_n\} \subseteq S - \{0\}$ with $g = s_1 + s_2 + \cdots + s_n$. The factorization is favorable if and only if $-s_i \notin \{s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n\}$.

**Proposition 2.5.** Let $\langle a_n \rangle_{n=1}^{\infty}$ be a sequence in $G$ and $S = \{\sum_{i=1}^{n} g_i \mid g_i \in T(i) \text{ for } 1 \leq i \leq n \text{ and } n \in \mathbb{N}\}$. If

1. every element of $S$ has only finitely many favorable factorizations in $S$;
2. if $a = \sum_{i=1}^{m} a_i$ for some $n, m \in \mathbb{N}$, then $a$ has no other favorable factorizations in $S$;

then the sequence $\langle a_n \rangle_{n=1}^{\infty}$ is a $T_{\Omega}$-sequence.

**Proof.** For any sequence $D = \langle d_n \rangle_{n=1}^{\infty} \in \zeta$ we have that $U(\langle d_n \rangle_{n=1}^{\infty}) \subseteq S$. So by (1) we have that for every $g \in S$ there exists a $k \in \mathbb{N}$ such that no favorable factorization of $g$ in $S$ has a factor in $T(k)$. Hence $g \notin U(\langle d_n \rangle_{n=1}^{\infty})$ and thus $\mathcal{F}_D$ generates a Hausdorff group topology.

Now choose $C = \{c_n\}_{n=1}^{\infty}$ and $D = \langle d_n \rangle_{n=1}^{\infty}$ in $\zeta$ with the property that for each $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that $c_k + m < d_m$ for all $m \geq N_k$. Suppose that $U(\langle d_n \rangle_{n=1}^{\infty})$ is open in the topology generated by $\mathcal{F}_C$. Then there exists a $k$ such that $U(\langle c_k \nabla_n \rangle_{n=1}^{\infty}) \subseteq U(\langle d_n \rangle_{n=1}^{\infty})$. We have that $a = \sum_{i=1}^{N_k} a_i c_k n_i \in U(\langle c_k \nabla_n \rangle_{n=1}^{\infty})$. But by (2) and the fact that $c_k N_k + N_k < b N_k$, we must conclude that $a \notin U(\langle d_n \rangle_{n=1}^{\infty})$. Hence the group topology generated by $\mathcal{F}_C$ is different from the group topology $\mathcal{F}_D$. By Lemma 2.3 we can find uncountably many different Hausdorff group topologies on $G$ with the property that $\langle a_n \rangle_{n=1}^{\infty}$ converges to zero.

**Example 2.6.** Let $\langle p^n \nabla_n \rangle_{n=1}^{\infty}$ be the sequence of powers of the prime $p$ in $\mathbb{Z}$. $\langle p^n \nabla_n \rangle_{n=1}^{\infty}$ is a $T_{\Omega}$-sequence.

**Example 2.7.** Let $k \in \mathbb{N}$ and let $\langle a_n \rangle_{n=1}^{\infty}$ be an increasing sequence in $\mathbb{Z}$ satisfying the inequality $a_{n+1}/a_n > n/k$ for all $n$. For $n > 2k$ we have that $\sum_{i=1}^{m} a_{n+i} < a_{n+m+1}$ for each $m \in \mathbb{N}$. Hence $\langle a_n \rangle_{n=1}^{\infty}$ is a $T_{\Omega}$-sequence.

**Example 2.8.** Let $Z \in \mathbb{Z}(p^\infty)$. The order of $Z$ is $p^n$ if $Z$ is a $p^n$-root of unity but not a $p^{n-1}$-root of unity. We denote the order of $Z$ by $O(Z)$. Now if $O(Z) = p^m$ and $O(w) = p^n$ and $m < n$ we have that $O(Zw) = p^n$. Let $\langle Z_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}(p^\infty)$ satisfying

$$O(Z^{p^i}) \geq p^{i+1}O(Z^{p^{i+1-i}}) \text{ for all } n \in \mathbb{N} \text{ and for } 0 \leq i < p.$$ (2.2)

By Proposition 2.5, $\langle Z_n \rangle_{n=1}^{\infty}$ is a $T_{\Omega}$-sequence.
Example 2.9. Consider \( \mathbb{R} \) as the direct sum of uncountably many copies of \( \mathbb{Q} \). If \( \langle r_n \rangle_{n=1}^\infty \) is any sequence of linearly independent real numbers, then \( \langle r_n \rangle_{n=1}^\infty \) is a \( T_\Omega \)-sequence.

We end this paper with a question. Does there exist a nontrivial sequence in a group \( G \) which is a \( T \)-sequence, but not a \( T_\Omega \)-sequence?

References


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