INTUITIONISTIC FUZZY IDEALS OF BCK-ALGEBRAS

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ABSTRACT. We consider the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigate some of their properties. We introduce the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigate some related properties.

Keywords and phrases. (Intuitionistic) fuzzy subalgebra, (intuitionistic) fuzzy ideal, upper (respectively, lower) \( t \)-level cut, homomorphism.

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1. Introduction. After the introduction of the concept of fuzzy sets by Zadeh [9] several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of “intuitionistic fuzzy set” was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. The first author (together with Hong, Kim, Kim, Meng, Roh, and Song) considered the fuzzification of ideals and subalgebras in BCK-algebras (cf. [3, 4, 5, 6, 7, 8]). In this paper, using the Atanassov’s idea, we establish the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigate some of their properties. We introduce the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigate some related properties.

2. Preliminaries. First we present the fundamental definitions. By a \( BCK\)-algebra we mean a nonempty set \( X \) with a binary operation \( * \) and a constant \( 0 \) satisfying the following conditions:

(I) \( ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0 \),

(II) \( (x \ast (x \ast y)) \ast y = 0 \),

(III) \( x \ast x = 0 \),

(IV) \( 0 \ast x = 0 \),

(V) \( x \ast y = 0 \) and \( y \ast x = 0 \) imply that \( x = y \)

for all \( x, y, z \in X \).

A partial ordering \( \leq \) on \( X \) can be defined by \( x \leq y \) if and only if \( x \ast y = 0 \). A nonempty subset \( S \) of a BCK-algebra \( X \) is called a subalgebra of \( X \) if \( x \ast y \in S \) whenever \( x, y \in S \). A nonempty subset \( I \) of a BCK-algebra \( X \) is called an ideal of \( X \) if

(i) \( 0 \in I \),

(ii) \( x \ast y \in I \) and \( y \in I \) imply that \( x \in I \) for all \( x, y \in X \).

By a fuzzy set \( \mu \) in a nonempty set \( X \) we mean a function \( \mu : X \rightarrow [0, 1] \), and the complement of \( \mu \), denoted by \( \bar{\mu} \), is the fuzzy set in \( X \) given by \( \bar{\mu}(x) = 1 - \mu(x) \) for all \( x \in X \). A fuzzy set \( \mu \) in a BCK-algebra \( X \) is called a fuzzy subalgebra of \( X \) if \( \mu(x \ast y) \geq \mu(x) \ast \mu(y) \).
min{\mu(x),\mu(y)} for all \(x,y \in X\). A fuzzy set \(\mu\) in a BCK-algebra \(X\) is called a fuzzy ideal of \(X\) if

(i) \(\mu(0) \geq \mu(x)\) for all \(x \in X\),

(ii) \(\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}\) for all \(x,y \in X\).

An intuitionistic fuzzy set (briefly, IFS) \(A\) in a nonempty set \(X\) is an object having the form

\[
A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\},
\]

(2.1)

where the functions \(\alpha_A : X \to [0,1]\) and \(\beta_A : X \to [0,1]\) denote the degree of membership and the degree of nonmembership, respectively, and

\[
0 \leq \alpha_A(x) + \beta_A(x) \leq 1 \quad \forall x \in X.
\]

(2.2)

An intuitionistic fuzzy set \(A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}\) in \(X\) can be identified to an ordered pair \((\alpha_A, \beta_A)\) in \(I^X \times I^X\). For the sake of simplicity, we shall use the symbol \(A = (\alpha_A, \beta_A)\) for the IFS \(A = \{(x, \alpha_A(x), \beta_A(x)) \mid x \in X\}\).

3. Intuitionistic fuzzy ideals. In what follows, let \(X\) denote a BCK-algebra unless otherwise specified.

**Definition 3.1.** An IFS \(A = (\alpha_A, \beta_A)\) in \(X\) is called an intuitionistic fuzzy subalgebra of \(X\) if it satisfies:

(IS1) \(\alpha_A(x \ast y) \geq \min\{\alpha_A(x), \alpha_A(y)\}\),

(IS2) \(\beta_A(x \ast y) \leq \max\{\beta_A(x), \beta_A(y)\}\),

for all \(x,y \in X\).

**Example 3.2.** Consider a BCK-algebra \(X = \{0, a, b, c\}\) with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \(A = (\alpha_A, \beta_A)\) be an IFS in \(X\) defined by

\[
\alpha_A(0) = \alpha_A(a) = \alpha_A(c) = 0.7 > 0.3 = \alpha_A(b),
\]

\[
\beta_A(0) = \beta_A(a) = \beta_A(c) = 0.2 < 0.5 = \beta_A(b).
\]

(3.1)

Then \(A = (\alpha_A, \beta_A)\) is an intuitionistic fuzzy subalgebra of \(X\).

**Proposition 3.3.** Every intuitionistic fuzzy subalgebra \(A = (\alpha_A, \beta_A)\) of \(X\) satisfies the inequalities \(\alpha_A(0) \geq \alpha_A(x)\) and \(\beta_A(0) \leq \beta_A(x)\) for all \(x \in X\).

**Proof.** For any \(x \in X\), we have

\[
\alpha_A(0) = \alpha_A(x \ast x) \geq \min\{\alpha_A(x), \alpha_A(x)\} = \alpha_A(x),
\]

\[
\beta_A(0) = \beta_A(x \ast x) \leq \max\{\beta_A(x), \beta_A(x)\} = \beta_A(x).
\]

(3.2)

This completes the proof.
**Definition 3.4.** An IFS \( A = (\alpha_A, \beta_A) \) in \( X \) is called an *intuitionistic fuzzy ideal* of \( X \) if it satisfies the following inequalities:

(IF1) \( \alpha_A(0) \geq \alpha_A(x) \) and \( \beta_A(0) \leq \beta_A(x) \),

(IF2) \( \alpha_A(x) \geq \min\{\alpha_A(x \ast y), \alpha_A(y)\} \),

(IF3) \( \beta_A(x) \leq \max\{\beta_A(x \ast y), \beta_A(y)\} \),

for all \( x, y \in X \).

**Example 3.5.** Let \( X = \{0, 1, 2, 3, 4\} \) be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
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<td>2</td>
<td>2</td>
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<td>0</td>
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<td>3</td>
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<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Define an IFS \( A = (\alpha_A, \beta_A) \) in \( X \) as follows:

\[
\alpha_A(0) = \alpha_A(2) = 1, \quad \alpha_A(1) = \alpha_A(3) = \alpha_A(4) = t, \\
\beta_A(0) = \beta_A(2) = 0, \quad \beta_A(1) = \beta_A(3) = \beta_A(4) = s,
\]

(3.3)

where \( t \in [0, 1] \), \( s \in [0, 1] \), and \( t + s \leq 1 \). By routine calculation we know that \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy ideal of \( X \).

**Lemma 3.6.** Let an IFS \( A = (\alpha_A, \beta_A) \) in \( X \) be an intuitionistic fuzzy ideal of \( X \). If the inequality \( x \ast y \leq z \) holds in \( X \), then

\[
\alpha_A(x) \geq \min\{\alpha_A(y), \alpha_A(z)\}, \quad \beta_A(x) \leq \max\{\beta_A(y), \beta_A(z)\}.
\]

(3.4)

**Proof.** Let \( x, y, z \in X \) be such that \( x \ast y \leq z \). Then \( (x \ast y) \ast z = 0 \), and thus

\[
\alpha_A(x) \geq \min\{\alpha_A((x \ast y) \ast z), \alpha_A(z)\}, \quad \alpha_A(x) \geq \min\{\alpha_A(x \ast y), \alpha_A(z)\}
\]

\[
= \min\{\min\{\alpha_A(x \ast y), \alpha_A(z)\}, \alpha_A(y)\}
\]

\[
= \min\{\alpha_A(y), \alpha_A(z)\},
\]

(3.5)

\[
\beta_A(x) \leq \max\{\beta_A((x \ast y) \ast z), \beta_A(z)\}, \quad \beta_A(x) \leq \max\{\beta_A(x \ast y), \beta_A(z)\}
\]

\[
= \max\{\max\{\beta_A(x \ast y), \beta_A(z)\}, \beta_A(y)\}
\]

\[
= \max\{\beta_A(y), \beta_A(z)\},
\]

this completes the proof. \( \square \)

**Lemma 3.7.** Let \( A = (\alpha_A, \beta_A) \) be an intuitionistic fuzzy ideal of \( X \). If \( x \leq y \) in \( X \), then

\[
\alpha_A(x) \geq \alpha_A(y), \quad \beta_A(x) \leq \beta_A(y),
\]

(3.6)

that is, \( \alpha_A \) is order-reserving and \( \beta_A \) is order-preserving.
Proof. Let \( x, y \in X \) be such that \( x \leq y \). Then \( x \ast y = 0 \) and so
\[
\begin{align*}
\alpha_A(x) & \geq \min \{ \alpha_A(x \ast y), \alpha_A(y) \} = \min \{ \alpha_A(0), \alpha_A(y) \} = \alpha_A(y), \\
\beta_A(x) & \leq \max \{ \beta_A(x \ast y), \beta_A(y) \} = \max \{ \beta_A(0), \beta_A(y) \} = \beta_A(y).
\end{align*}
\] (3.7)
This completes the proof.

**Theorem 3.8.** If \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy ideal of \( X \), then for any \( x, a_1, a_2, \ldots, a_n \in X \), \(( (x \ast a_1) \ast a_2) \ast \cdots \ast a_n = 0 \) implies
\[
\begin{align*}
\alpha_A(x) & \geq \min \{ \alpha_A(a_1), \alpha_A(a_2), \ldots, \alpha_A(a_n) \}, \\
\beta_A(x) & \leq \max \{ \beta_A(a_1), \beta_A(a_2), \ldots, \beta_A(a_n) \}.
\end{align*}
\] (3.8)

Proof. Using induction on \( n \) and Lemmas 3.6 and 3.7, the proof is straightforward.

The converse of Theorem 3.9 may not be true. For example, the intuitionistic fuzzy subalgebra \( A = (\alpha_A, \beta_A) \) in Example 3.2 is not an intuitionistic fuzzy ideal of \( X \) since
\[
\beta_A(b) = 0.5 > 0.2 = \min \{ \beta_A(b \ast a), \beta_A(a) \}.
\] (3.11)
We now give a condition for an intuitionistic fuzzy subalgebra to be an intuitionistic fuzzy ideal.

**Theorem 3.10.** Let \( A = (\alpha_A, \beta_A) \) be an intuitionistic fuzzy subalgebra of \( X \) such that
\[
\begin{align*}
\alpha_A(x) & \geq \min \{ \alpha_A(y), \alpha_A(z) \}, \\
\beta_A(x) & \leq \max \{ \beta_A(y), \beta_A(z) \}
\end{align*}
\] (3.12)
for all \( x, y, z \in X \) satisfying the inequality \( x \ast y \leq z \). Then \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy ideal of \( X \).

Proof. Let \( A = (\alpha_A, \beta_A) \) be an intuitionistic fuzzy subalgebra of \( X \). Recall that \( \alpha_A(0) \geq \alpha_A(x) \) and \( \beta_A(0) \leq \beta_A(x) \) for all \( X \). Since \( x \ast (x \ast y) \leq y \), it follows from the hypothesis that
\[
\begin{align*}
\alpha_A(x) & \geq \min \{ \alpha_A(x \ast y), \alpha_A(y) \}, \\
\beta_A(x) & \leq \max \{ \beta_A(x \ast y), \beta_A(y) \}.
\end{align*}
\] (3.13)
Hence \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy ideal of \( X \).
**Lemma 3.11.** An IFS $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$ if and only if the fuzzy sets $\alpha_A$ and $\beta_A$ are fuzzy ideals of $X$.

**Proof.** Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of $X$. Clearly, $\alpha_A$ is a fuzzy ideal of $X$. For every $x, y \in X$, we have

$$
\beta_A(x) = 1 - \beta_A(x) \geq 1 - \max \{\beta_A(x \ast y), \beta_A(y)\}
= \min \{1 - \beta_A(x \ast y), 1 - \beta_A(y)\}
= \min \{\beta_A(x \ast y), \beta_A(y)\},
$$

(3.14)

Hence $\beta_A$ is a fuzzy ideal of $X$.

Conversely, assume that $\alpha_A$ and $\beta_A$ are fuzzy ideals of $X$. For every $x, y \in X$, we get

$$
\alpha_A(0) \geq \alpha_A(x), \quad 1 - \beta_A(0) = \beta_A(0) \geq \beta_A(x) = 1 - \beta_A(x),
$$

(3.15)

that is, $\beta_A(0) \leq \beta_A(x)$; $\alpha_A(x) \geq \min \{\alpha_A(x \ast y), \alpha_A(y)\}$ and

$$
1 - \beta_A(x) = \beta_A(x) \geq \min \{\beta_A(x \ast y), \beta_A(y)\}
= \min \{1 - \beta_A(x \ast y), 1 - \beta_A(y)\}
= 1 - \max \{\beta_A(x \ast y), \beta_A(y)\},
$$

(3.16)

that is, $\beta_A(x) \leq \max \{\beta_A(x \ast y), \beta_A(y)\}$. Hence $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$.

**Theorem 3.12.** Let $A = (\alpha_A, \beta_A)$ be an IFS in $X$. Then $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$ if and only if $\Box A = (\alpha_A, \check{\alpha}_A)$ and $\Diamond A = (\check{\beta}_A, \beta_A)$ are intuitionistic fuzzy ideals of $X$.

**Proof.** If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$, then $\alpha_A = \check{\alpha}_A$ and $\beta_A$ are fuzzy ideals of $X$ from Lemma 3.11, hence $\Box A = (\alpha_A, \check{\alpha}_A)$ and $\Diamond A = (\check{\beta}_A, \beta_A)$ are intuitionistic fuzzy ideals of $X$. Conversely, if $\Box A = (\alpha_A, \check{\alpha}_A)$ and $\Diamond A = (\check{\beta}_A, \beta_A)$ are intuitionistic fuzzy ideals of $X$, then the fuzzy sets $\alpha_A$ and $\beta_A$ are fuzzy ideals of $X$, hence $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$.

For any $t \in [0, 1]$ and a fuzzy set $\mu$ in a nonempty set $X$, the set

$$
U(\mu; t) = \{x \in X \mid \mu(x) \geq t\}
$$

(3.17)

is called an upper $t$-level cut of $\mu$ and the set

$$
L(\mu; t) = \{x \in X \mid \mu(x) \leq t\}
$$

(3.18)

is called a lower $t$-level cut of $\mu$.

**Theorem 3.13.** An IFS $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of $X$ if and only if for all $s, t \in [0, 1]$, the sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are either empty or ideals of $X$. 
\textbf{Proof.} Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of $X$ and $U(\alpha_A; t) \neq \emptyset \neq L(\beta_A; s)$ for any $s, t \in [0, 1]$. It is clear that $0 \in U(\alpha_A; t) \cap L(\beta_A; s)$ since $\alpha_A(0) \geq t$ and $\beta_A(0) \leq s$. Let $x, y \in X$ be such that $x \ast y \in U(\alpha_A; t)$ and $y \in U(\alpha_A; t)$. Then $\alpha_A(x \ast y) \geq t$ and $\alpha_A(y) \geq t$. It follows that

$$\alpha_A(x) \geq \min \{\alpha_A(x \ast y), \alpha_A(y)\} \geq t$$

(3.19)

so that $x \in U(\alpha_A; t)$. Hence $U(\alpha_A; t)$ is an ideal of $X$. Now let $x, y \in X$ be such that $x \ast y \in L(\beta_A; s)$ and $y \in L(\beta_A; s)$. Then $\beta_A(x \ast y) \leq s$ and $\beta_A(y) \leq s$, which imply that

$$\beta_A(x) \leq \max \{\beta_A(x \ast y), \beta_A(y)\} \leq s.$$ 

(3.20)

Thus $x \in L(\beta_A; s)$, and therefore $L(\beta_A; s)$ is an ideal of $X$. Conversely, assume that for each $t, s \in [0, 1]$, the sets $U(\alpha_A; t)$ and $L(\beta_A; s)$ are either empty or ideals of $X$. For any $x \in X$, let $\alpha_A(x) = t$ and $\beta_A(x) = s$. Then $x \in U(\alpha_A; t) \cap L(\beta_A; s)$, and so $U(\alpha_A; t) \neq \emptyset \neq L(\beta_A; s)$. Since $U(\alpha_A; t)$ and $L(\beta_A; s)$ are ideals of $X$, therefore $0 \in U(\alpha_A; t) \cap L(\beta_A; s)$. Hence $\alpha_A(0) \geq t = \alpha_A(x)$ and $\beta_A(0) \leq s = \beta_A(x)$ for all $x \in X$. If there exist $x', y' \in X$ such that $\alpha_A(x') < \min \{\alpha_A(x' \ast y'), \alpha_A(y')\}$, then by taking

$$t_0 = \frac{1}{2} (\alpha_A(x') + \min \{\alpha_A(x' \ast y'), \alpha_A(y')\}),$$

(3.21)

we have

$$\alpha_A(x') < t_0 < \min \{\alpha_A(x' \ast y'), \alpha_A(y')\}.$$ 

(3.22)

Hence $x' \notin U(\alpha_A; t_0)$, $x' \ast y' \in U(\alpha_A; t_0)$ and $y' \in (\alpha_A; t_0)$, that is, $U(\alpha_A; t_0)$ is not an ideal of $X$, which is a contradiction. Finally, assume that there exist $a, b \in X$ such that

$$\beta_A(a) > \max \{\beta_A(a \ast b), \beta_A(b)\}.$$ 

(3.23)

Taking $s_0 := (1/2)(\beta_A(a) + \max \{\beta_A(a \ast b), \beta_A(b)\})$, then

$$\max \{\beta_A(a \ast b), \beta_A(b)\} < s_0 < \beta_A(a).$$ 

(3.24)

Therefore $a \ast b \in L(\beta_A; s_0)$ and $b \in L(\beta_A; s_0)$, but $a \notin L(\beta_A; s_0)$, which is a contradiction, this completes the proof. 

Let $\Lambda$ be a nonempty subset of $[0, 1]$.

\textbf{Theorem 3.14.} Let $\{I_t \mid t \in \Lambda\}$ be a collection of ideals of $X$ such that

(i) $X = \cup_{t \in \Lambda} I_t$,

(ii) $s > t$ if and only if $I_s \subseteq I_t$ for all $s, t \in \Lambda$.

Then an IFSA $=(\alpha_A, \beta_A)$ in $X$ defined by

$$\alpha_A(x) := \sup \{t \in \Lambda \mid x \in I_t\}, \quad \beta_A(x) := \inf \{t \in \Lambda \mid x \in I_t\}$$

(3.25)

for all $x \in X$ is an intuitionistic fuzzy ideal of $X$.

\textbf{Proof.} According to Theorem 3.13, it is sufficient to show that $U(\alpha_A; t)$ and $L(\beta_A; s)$ are ideals of $X$ for every $t \in [0, \alpha_A(0)]$ and $s \in [\beta_A(0), 1]$. In order to prove
that $U(\alpha_A; t)$ is an ideal of $X$, we divide the proof into the following two cases:

(i) $t = \sup \{ q \in \Lambda \mid q < t \}$,
(ii) $t \neq \sup \{ q \in \Lambda \mid q < t \}$.

Case (i) implies that

$$ x \in U(\alpha_A; t) \iff x \in I_q \quad \forall q < t \iff x \in \cap_{q < t} I_q, \quad (3.26) $$

so that $U(\alpha_A; t) = \cap_{q < t} I_q$, which is an ideal of $X$. For the case (ii), we claim that $U(\alpha_A; t) = \cup_{q \geq t} I_q$. If $x \in \cup_{q \geq t} I_q$, then $x \in I_q$ for some $q \geq t$. It follows that $\alpha_A(x) \geq q \geq t$, so that $x \in U(\alpha_A; t)$. This shows that $\cup_{q \geq t} I_q \subseteq U(\alpha_A; t)$. Now assume that $x \notin \cup_{q \geq t} I_q$. Then $x \notin I_q$ for all $q \geq t$. Since $t \neq \sup \{ q \in \Lambda \mid q < t \}$, there exists $\varepsilon > 0$ such that $(t - \varepsilon, t) \cap \Lambda = \emptyset$. Hence $x \notin I_q$ for all $q > t - \varepsilon$, which means that if $x \in I_q$, then $q < t - \varepsilon$. Thus $\alpha_A(x) \leq t - \varepsilon < t$, and so $x \notin U(\alpha_A; t)$. Therefore $U(\alpha_A; t) \subseteq \cup_{q < t} I_q$, and thus $U(\alpha_A; t) = \cup_{q \geq t} I_q$ which is an ideal of $X$. Next we prove that $L(\beta_A; s)$ is an ideal of $X$. We consider the following two cases:

(iii) $s = \inf \{ r \in \Lambda \mid s < r \}$,
(iv) $s = \inf \{ r \in \Lambda \mid s = r \}$.

For the case (iii), we have

$$ x \in L(\beta_A; s) \iff x \in I_r \quad \forall s < r \iff x \in \cap_{s < r} I_r, \quad (3.27) $$

and hence $L(\beta_A; s) = \cap_{s < r} I_r$ which is an ideal of $X$. For the case (iv) there exists $\varepsilon > 0$ such that $(s, s + \varepsilon) \cap \Lambda = \emptyset$. We will show that $L(\beta_A; s) = \cup_{s \geq r} I_r$. If $x \in \cup_{s \geq r} I_r$, then $x \in I_r$ for some $r \leq s$. It follows that $\beta_A(x) \leq r \leq s$ so that $x \in L(\beta_A; s)$. Hence $\cup_{s \geq r} I_r \subseteq L(\beta_A; s)$. Conversely, if $x \notin \cup_{s \geq r} I_r$, then $x \notin I_r$ for all $r \leq s$, which implies that $x \notin I_r$ for all $r < s + \varepsilon$, that is, if $x \in I_r$, then $r \geq s + \varepsilon$. Thus $\beta_A(x) \geq s + \varepsilon > s$, that is, $x \notin L(\beta_A; s)$. Therefore $L(\beta_A; s) \subseteq \cup_{s \geq r} I_r$ and consequently $L(\beta_A; s) = \cup_{s \geq r} I_r$ which is an ideal of $X$. This completes the proof.

A mapping $f : X \to Y$ of BCK-algebras is called a homomorphism if $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCK-algebras, then $f(0) = 0$. Let $f : X \to Y$ be a homomorphism of BCK-algebras. For any $\text{IFS} = (\alpha_A, \beta_A)$ in $Y$, we define a new $\text{IFS}^f = (\alpha^f_A, \beta^f_A)$ in $X$ by

$$ \alpha^f_A(x) := \alpha_A(f(x)), \quad \beta^f_A(x) := \beta_A(f(x)) \quad \forall x \in X. \quad (3.28) $$

**Theorem 3.15.** Let $f : X \to Y$ be a homomorphism of BCK-algebras. If an $\text{IFS} = (\alpha_A, \beta_A)$ in $Y$ is an intuitionistic fuzzy ideal of $Y$, then an $\text{IFS}^f = (\alpha^f_A, \beta^f_A)$ in $X$ is an intuitionistic fuzzy ideal of $X$.

**Proof.** We first have that

$$ \alpha^f_A(x) = \alpha_A(f(x)) \leq \alpha_A(0) = \alpha_A(f(0)) = \alpha^f_A(0), $$

$$ \beta^f_A(x) = \beta_A(f(x)) \geq \beta_A(0) = \beta_A(f(0)) = \beta^f_A(0) \quad (3.29) $$

for all $x \in X$. Let $x, y \in X$. Then
\[
\begin{align*}
\min \{\alpha_A(x \ast y), \alpha_A(y)\} &= \min \{\alpha_A(f(x \ast y)), \alpha_A(f(y))\} \\
&= \min \{\alpha_A(f(x) \ast f(y)), \alpha_A(f(y))\} \\
&\leq \alpha_A(f(x)) = \alpha_A^f(x), \\
\max \{\beta_A(x \ast y), \beta_A(y)\} &= \max \{\beta_A(f(x \ast y)), \beta_A(f(y))\} \\
&= \max \{\beta_A(f(x) \ast f(y)), \beta_A(f(y))\} \\
&\geq \beta_A(f(x)) = \beta_A^f(x).
\end{align*}
\]
(3.30)

Hence \( A^f = (\alpha_A^f, \beta_A^f) \) is an intuitionistic fuzzy ideal of \( X \).

If we strengthen the condition of \( f \), then we can construct the converse of Theorem 3.15 as follows.

**Theorem 3.16.** Let \( f : X \to Y \) be an epimorphism of BCK-algebras and let \( A = (\alpha_A, \beta_A) \) be an IFS in \( Y \). If \( A^f = (\alpha_A^f, \beta_A^f) \) is an intuitionistic fuzzy ideal of \( X \), then \( A = (\alpha_A, \beta_A) \) is an intuitionistic fuzzy ideal of \( Y \).

**Proof.** For any \( x \in Y \), there exists \( a \in X \) such that \( f(a) = x \). Then
\[
\begin{align*}
\alpha_A(x) &= \alpha_A(f(a)) = \alpha_A^f(a) \leq \alpha_A^f(0) = \alpha_A(f(0)) = \alpha_A(0), \\
\beta_A(x) &= \beta_A(f(a)) = \beta_A^f(a) \geq \beta_A^f(0) = \beta_A(f(0)) = \beta_A(0).
\end{align*}
\]
(3.31)

Let \( x, y \in Y \). Then \( f(a) = x \) and \( f(b) = y \) for some \( a, b \in X \). It follows that
\[
\begin{align*}
\alpha_A(x) &= \alpha_A(f(a)) = \alpha_A^f(a) \\
&\geq \min \{\alpha_A^f(a \ast b), \alpha_A^f(b)\} \\
&= \min \{\alpha_A(f(a \ast b)), \alpha_A(f(b))\} \\
&= \min \{\alpha_A(f(a) \ast f(b)), \alpha_A(f(b))\} \\
&= \min \{\alpha_A(x \ast y), \alpha_A(y)\}, \\
\beta_A(x) &= \beta_A(f(a)) = \beta_A^f(a) \\
&\leq \max \{\beta_A^f(a \ast b), \beta_A^f(b)\} \\
&= \max \{\beta_A(f(a \ast b)), \beta_A(f(b))\} \\
&= \max \{\beta_A(f(a) \ast f(b)), \beta_A(f(b))\} \\
&= \max \{\beta_A(x \ast y), \beta_A(y)\}.
\end{align*}
\]
(3.32)

This completes the proof.

Let \( \text{IF}(X) \) be the family of all intuitionistic fuzzy ideals of \( X \) and let \( t \in [0,1] \). Define binary relations \( U^t \) and \( L^t \) on \( \text{IF}(X) \) as follows:
\[
(A, B) \in U^t \iff U(\alpha_A; t) = U(\alpha_B; t), \quad (A, B) \in L^t \iff L(\beta_A; t) = L(\beta_B; t), \quad (3.33)
\]
respectively, for \( A = (\alpha_A, \beta_A) \) and \( B = (\alpha_B, \beta_B) \) in \( \text{IF}(X) \). Then clearly \( U^t \) and \( L^t \) are
equivalence relations on $\text{IF}(X)$. For any $A = (\alpha_A, \beta_A) \in \text{IF}(X)$, let $[A]_{U^t}$ (respectively, $[A]_{L^t}$) denote the equivalence class of $A$ modulo $U^t$ (respectively, $L^t$), and denote by $\text{IF}(X)/U^t$ (respectively, $\text{IF}(X)/L^t$) the system of all equivalence classes modulo $U^t$ (respectively, $L^t$); so

$$\text{IF}(X)/U^t := \{[A]_{U^t} \mid A = (\alpha_A, \beta_A) \in \text{IF}(X)\}, \quad (3.34)$$

respectively,

$$\text{IF}(X)/L^t := \{[A]_{L^t} \mid A = (\alpha_A, \beta_A) \in \text{IF}(X)\}. \quad (3.35)$$

Now let $I(X)$ denote the family of all ideals of $X$ and let $t \in [0, 1]$. Define maps $f_t$ and $g_t$ from $\text{IF}(X)$ to $I(X) \cup \{\emptyset\}$ by $f_t(A) = U(\alpha_A; t)$ and $g_t(A) = L(\beta_A; t)$, respectively, for all $A = (\alpha_A, \beta_A) \in \text{IF}(X)$. Then $f_t$ and $g_t$ are clearly well defined.

**Theorem 3.17.** For any $t \in (0, 1)$ the maps $f_t$ and $g_t$ are surjective from $\text{IF}(X)$ to $I(X) \cup \{\emptyset\}$.

**Proof.** Let $t \in (0, 1)$. Note that $0_+ = (0, 1)$ is in $\text{IF}(X)$, where $0$ and $1$ are fuzzy sets in $X$ defined by $0(x) = 0$ and $1(x) = 1$ for all $x \in X$. Obviously $f_t(0_+) = U(0; t) = \emptyset = L(1; t) = g_t(0_+)$. Let $G(\neq \emptyset) \in I(X)$. For $G_- = (\chi_G, \bar{\chi}_G) \in \text{IF}(X)$, we have $f_t(G_-) = U(\chi_G; t) = G$ and $g_t(G_-) = L(\bar{\chi}_G; t) = G$. Hence $f_t$ and $g_t$ are surjective. □

**Theorem 3.18.** The quotient sets $\text{IF}(X)/U^t$ and $\text{IF}(X)/L^t$ are equipotent to $I(X) \cup \{\emptyset\}$ for every $t \in (0, 1)$.

**Proof.** For $t \in (0, 1)$ let $f_t^*$ (respectively, $g_t^*$) be a map from $\text{IF}(X)/U^t$ (respectively, $\text{IF}(X)/L^t$) to $I(X) \cup \{\emptyset\}$ defined by $f_t^*([A]_{U^t}) = f_t(A)$ (respectively, $g_t^*([A]_{L^t}) = g_t(A)$) for all $A = (\alpha_A, \beta_A) \in \text{IF}(X)$. If $U(\alpha_A; t) = U(\alpha_B; t)$ and $L(\beta_A; t) = L(\beta_B; t)$ for $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ in $\text{IF}(X)$, then $(A, B) \in U^t$ and $(A, B) \in L^t$; hence $[A]_{U^t} = [B]_{U^t}$ and $[A]_{L^t} = [B]_{L^t}$. Therefore the maps $f_t^*$ and $g_t^*$ are injective. Now let $G(\neq \emptyset) \in I(X)$. For $G_- = (\chi_G, \bar{\chi}_G) \in \text{IF}(X)$, we have

$$f_t^*([G_-]_{U^t}) = f_t(G_-) = U(\chi_G; t) = G,$$

$$g_t^*([G_-]_{L^t}) = g_t(G_-) = L(\bar{\chi}_G; t) = G. \quad (3.36)$$

Finally, for $0_- = (0, 1) \in \text{IF}(X)$ we get

$$f_t^*([0_-]_{U^t}) = f_t(0_-) = U(0; t) = \emptyset,$$

$$g_t^*([0_-]_{L^t}) = g_t(0_-) = L(0; t) = \emptyset. \quad (3.37)$$

This shows that $f_t^*$ and $g_t^*$ are surjective. This completes the proof. □

For any $t \in [0, 1]$, we define another relation $R^t$ on $\text{IF}(X)$ as follows:

$$(A, B) \in R^t \iff U(\alpha_A; t) \cap L(\beta_A; t) = U(\alpha_B; t) \cap L(\beta_B; t) \quad (3.38)$$
for any $A = (\alpha_A, \beta_A), B = (\alpha_B, \beta_B) \in \text{IF}(X)$. Then the relation $R^t$ is also an equivalence relation on $\text{IF}(X)$.

**Theorem 3.19.** For any $t \in (0, 1)$, the map $\phi_t : \text{IF}(X) \to I(X) \cup \{\emptyset\}$ defined by $\phi_t(A) = f_t(A) \cap g_t(A)$ for each $A = (\alpha_A, \beta_A) \in \text{IF}(X)$ is surjective.

**Proof.** Let $t \in (0, 1)$. For $0_\sim = (0, 1) \in \text{IF}(X)$,

$$\phi_t(0_\sim) = f_t(0_\sim) \cap g_t(0_\sim) = U(0_t) \cap L(1_t) = \emptyset .$$

(3.39)

For any $H \in \text{IF}(X)$, there exists $H_\sim = (\chi_H, \tilde{\chi}_H) \in \text{IF}(X)$ such that

$$\phi_t(H_\sim) = f_t(H_\sim) \cap g_t(H_\sim) = U(\chi_H; t) \cap L(\tilde{\chi}_H; t) = H .$$

(3.40)

This completes the proof. \qed

**Theorem 3.20.** For any $t \in (0, 1)$, the quotient set $\text{IF}(X)/R^t$ is equipotent to $I(X) \cup \{\emptyset\}$.

**Proof.** Let $t \in (0, 1)$ and let $\phi_t^* : \text{IF}(X)/R^t \to I(X) \cup \{\emptyset\}$ be a map defined by $\phi_t^*([A]_{R^t}) = \phi_t(A)$ for all $[A]_{R^t} \in \text{IF}(X)/R^t$. If $\phi_t^*([A]_{R^t}) = \phi_t^*([B]_{R^t})$ for any $[A]_{R^t}, [B]_{R^t} \in \text{IF}(X)/R^t$, then $f_t(A) \cap g_t(A) = f_t(B) \cap g_t(B)$, that is, $U(\alpha_A; t) \cap L(\beta_A; t) = U(\alpha_B; t) \cap L(\beta_B; t)$, hence $(A, B) \in R^t$. It follows that $[A]_{R^t} = [B]_{R^t}$ so that $\phi_t^*$ is injective. For $0_\sim = (0, 1) \in \text{IF}(X)$,

$$\phi_t^*([0_\sim]_{R^t}) = \phi_t(0_\sim) = f_t(0_\sim) \cap g_t(0_\sim) = U(0_t) \cap L(1_t) = \emptyset .$$

(3.41)

If $H \in \text{IF}(X)$, then for $H_\sim = (\chi_H, \tilde{\chi}_H) \in \text{IF}(X)$, we have

$$\phi_t^*([H_\sim]_{R^t}) = \phi(H_\sim) = f_t(H_\sim) \cap g_t(H_\sim) = U(\chi_H; t) \cap L(\tilde{\chi}_H; t) = H .$$

(3.42)

Hence $\phi_t^*$ is surjective, this completes the proof. \qed

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