ON SOME TOPOLOGICAL PROPERTIES OF GENERALIZED DIFFERENCE SEQUENCE SPACES

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ABSTRACT. We obtain some topological results of the sequence spaces $\Delta^m(X)$, where $\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$, $(m \in \mathbb{N})$, and $X$ is any sequence space. We compute the $p\alpha$, $p\beta$, and $p\gamma$-duals of $l_\infty$, $c$, and $c_0$ and we investigate the $N$-(or null) dual of the sequence spaces $\Delta^m(l_\infty), \Delta^m(c)$, and $\Delta^m(c_0)$. Also we show that any matrix map from $\Delta^m(l_\infty)$ into a BK-space which does not contain any subspace isomorphic to $\Delta^m(l_\infty)$ is compact.

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1. Introduction. $w$ denotes the space of all scalar sequences and any subspace of $w$ is called a sequence space. The following sequence spaces will be used in what follows:

- $l_\infty$, the space of all bounded scalar sequences;
- $c$, the space of all convergent scalar sequences;
- $c_0$, the space of all null scalar sequences;
- $l_1$, the space of all absolutely 1-summable scalar sequences;
- $s$, the space of all real sequences;
- $s_0$, the space of all statistically convergent sequences of real numbers;
- $\Delta^m(l_\infty)$, the space of all $\Delta^m$-bounded scalar sequences;
- $\Delta^m(c)$, the space of all $\Delta^m$-convergent scalar sequences;
- $\Delta^m(c_0)$, the space of all $\Delta^m$-null scalar sequences;
- $\Delta^m(s_0)$, the space of all $\Delta^m$-statistically convergent sequences of real numbers.

It is known that $l_\infty$, $c$, and $c_0$ are $B$-spaces (Banach spaces) with their usual norm $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N} = \{1,2,\ldots\}$. The sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$, $c_0(\Delta^m)$ have been introduced by Et and Çolak [1]. These sequence spaces are BK-spaces (Banach coordinate spaces) with norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty,$$

where $m \in \mathbb{N}$, $\Delta^m x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$, and so

$$\Delta^m x_k = \sum_{\nu=0}^{m} (-1)^{\nu} \binom{m}{\nu} x_{k+\nu}. \quad (1.2)$$
For convenience we denote these spaces \( \Delta^m(l_\infty), \Delta^m(c), \) and \( \Delta^m(c_0) \) and call \( \Delta^m \) bounded, \( \Delta^m \)-convergent, and \( \Delta^m \)-null sequences, respectively. The operators
\[
\Delta^{(m)}, \quad \sum^{(m)} : w \to w
\]
are defined by
\[
\Delta^{(1)} x_k = x_k - x_{k-1}, \quad \sum^{(1)} x_k = \sum_{j=0}^{k} x_j, \quad (k = 0, 1, \ldots),
\]
\[
\Delta^{(m)} = \Delta^{(1)}_0 \Delta^{(m-1)}, \quad \sum^{(m)} = \sum_0^{(m-1)}, \quad (m \geq 2),
\]
and
\[
\sum^{(m)}_0 \Delta^{(m)} = \Delta^{(m)}_0 \sum^{(m)} = \text{id},
\]
the identity on \( w \) (see [4]).

For any subset \( X \) of \( w \) let
\[
\Delta^m(X) = \{ x = (x_k) : (\Delta^m x_k) \in X \}.
\]

Now we define
\[
\Delta^{(m)} x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k-v}.
\]

It is trivial that \( (\Delta^m x_k) \in X \) if and only if \( (\Delta^{(m)} x_k) \in X \), for \( X = l_\infty, c \) or \( c_0 \). In [4], Malkowsky and Parashar also showed that the sequence spaces \( \Delta^m(l_\infty), \Delta^m(c), \) and \( \Delta^m(c_0) \) are also BK-spaces with norm
\[
\|x\|_{\Delta^1} = \sup_k |\Delta^{(m)} x_k|.
\]

It is trivial that the norms (1.1) and (1.8) are equivalent. Obviously
\[
\Delta^{(m)} : \Delta^{(m)}(X) \to X, \quad \Delta^{(m)} x = y = (\Delta^{(m)} x_k),
\]
\[
\sum^{(m)} : X \to \Delta^{(m)}(X), \quad \sum^{(m)} x = y = \left( \sum^{(m)} x_k \right)
\]
are isometric isomorphism, for \( X = l_\infty, c \) or \( c_0 \).

Hence \( \Delta^m(l_\infty), \Delta^m(c), \) and \( \Delta^m(c_0) \) are isometrically isomorphic to \( l_\infty, c \), and \( c_0 \), respectively. Thus \( l_1 \) is continuous dual of \( \Delta^m(c) \) and \( \Delta^m(c_0) \).

Throughout the paper, we write \( \sum_k \) for \( \sum_{k=1}^{\infty} \) and \( \lim_n \) for \( \lim_{n \to \infty} \).

Let \( A = (a_{nk}) \) be an infinite matrix of complex numbers. Let \( E \) and \( F \) be BK-spaces. We write \( Ax = (A_n(x)) \) if \( A_n(x) = \sum_k a_{nk} x_k \) converges for each \( n \in \mathbb{N} \). If \( Ax = (A_n(x)) \in E \) for each \( x = (x_k) \in F \), then we say that \( A \) defines a matrix map from \( F \) into \( E \) and we denote it by \( A : F \to E \). By \( (F, E) \) we mean the class of matrices \( A \) such that \( A : F \to E \). We denote the set \( \{ x \in w : Ax \text{ exists and } Ax \in E \} \) by \( E_A \). Note that \( A \) is a matrix map from \( F \) into \( E \) if and only if \( F \subseteq E_A \). From now on, \( E \) unless specified shall denote a BK-space.
In $B$-space $E$, the following statements are equivalent (see [5]).

(i) $\sum x_n$ is unconditionally convergent.

(ii) $\sum x_n$ is weakly subseries convergent; that is, weak $\lim_n \sum_{j=1}^{n} x_{k_j}$ exists for each increasing sequence $(k_n)$ of positive integers.

(iii) $\sum x_n$ is subseries convergent; that is, norm $\lim_n \sum_{j=1}^{n} x_{k_j}$ exists with $(k_n)$ above.

(iv) $\sum x_n$ is bounded multiplier convergent; that is, $\sum x_n t_n$ exists for each sequence $t = (t_n)$ of bounded scalars.

2. Some properties of $\Delta^m(X)$. In this section, we will give some properties of $\Delta^m(X)$.

**Theorem 2.1.** Let $X$ be a vector space and let $A \subset X$. If $A$ is a convex set, then $\Delta^m(A)$ is a convex set in $\Delta^m(X)$.

**Proof.** Let $x, y \in \Delta^m(A)$, then $\Delta^m x, \Delta^m y \in A$. Since $\Delta^m$ is linear, we have

$$\lambda \Delta^m x + (1 - \lambda) \Delta^m y = \Delta^m (\lambda x + (1 - \lambda) y), \quad (0 \leq \lambda \leq 1).$$

(2.1)

Since $A$ is convex ($\lambda \Delta^m x + (1 - \lambda) \Delta^m y) \in A$ and so $(\lambda x + (1 - \lambda) y) \in \Delta^m(A)$,

(0 $\leq \lambda \leq 1$).

**Lemma 2.2.** Let $m$ be a positive integer. Then

(i) $\Delta^m(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} \Delta^m(A_n)$,

(ii) $\Delta^m(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} \Delta^m(A_n)$.

The proof is clear. \hfill $\Box$

**Lemma 2.3.** Let $X$ be a Banach space and let $A \subset X$. Then

(i) If $A$ is nowhere dense in $X$, then $\Delta^m(A)$ is nowhere dense in $\Delta^m(X)$.

(ii) If $A$ is dense in $X$, then $\Delta^m(A)$ is dense in $\Delta^m(X)$.

(iii) $\Delta^m(w) = w$, where $m$ is a positive integer.

**Proof.** (i) Suppose that $\overline{A} = \emptyset$, but $\overline{\Delta^m(A)} \neq \emptyset$. Then $\overline{\Delta^m(A)}$ contains no neighborhood of $B(a) \subset \overline{\Delta^m(A)}$, where $B(a)$ is a neighborhood (or open ball) of center $a$ and radius $r$. Hence $a \in B(a) \subset \overline{\Delta^m(A)} = \overline{\Delta^m(A)}$. This implies that $\Delta^m(a) \in \overline{\Delta^m(A)}$. So $B(\Delta^m(a)) \cap A \neq \emptyset$. On the other hand, $B(\Delta^m(a)) \cap A \subset \overline{A}$. This contradicts to $\overline{A} = \emptyset$. Hence $\overline{\Delta^m(A)} = \emptyset$.

(ii) and (iii) are trivial. \hfill $\Box$

**Theorem 2.4.** (i) The set $\Delta^m(s_0)$ is dense in the space $s$.

(ii) The set $\Delta^m(s_0)$ is a set of the first Baire category in the space $s$.

(iii) The set $s-\Delta^m(s_0)$ is a set of the second Baire category in the space $s$.

**Proof.** The proof follows from [6, Theorem 3.1], Lemmas 2.2, and 2.3, we recall that the complement $M^c$ of a meager (or of the first category) subset $M$ of a complete metric space $X$ is nonmeager (or of the second category). \hfill $\Box$

**Theorem 2.5.** $l_\infty \cap \Delta^m(c) = l_\infty \cap \Delta^m(c_0)$. 

This implies that $Hence a \in l_\infty \cap \Delta^m(c_0)$. Now let $x \in l_\infty \cap \Delta^m(c)$, then $x \in l_\infty$ and $\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1} \to 0$, $(k \to \infty)$, $\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1} = l + \varepsilon_k$. This implies that

$$l = n^{-1}\Delta^{m-1}x_1 - n^{-1}\Delta^{m-1}x_{n+1} + n^{-1}\sum_{k=1}^{n} \varepsilon_k.$$  

(2.2)

This yields $l = 0$ and $x \in l_\infty \cap \Delta^m(c_0)$.

\[\Box\]

3. Dual spaces. In this section, we give the $N$-dual (null dual) of the sequence spaces $\Delta^m(l_\infty)$, $\Delta^m(c)$, and $\Delta^m(c_0)$ and the $p\alpha$, $p\beta$, and $p\gamma$-duals of the sequence spaces of $l_\infty$, $c$, and $c_0$.

**Definition 3.1.** Let $X$ be a sequence space and define

$$X^\alpha = \left\{ a = (a_k) : \sum_k |a_kx_k| < \infty, \forall x \in X \right\},$$

$$X^\beta = \left\{ a = (a_k) : \sum_k a_kx_k \text{ is convergent }, \forall x \in X \right\},$$

$$X^\gamma = \left\{ a = (a_k) : \sup_n \left| \sum_k a_kx_k \right| < \infty, \forall x \in X \right\},$$

$$X^N = \left\{ a = (a_k) : \lim_k a_kx_k = 0, \forall x \in X \right\},$$

(3.1)

then $X^\alpha, X^\beta, X^\gamma,$ and $X^N$ are called the $\alpha$, $\beta$, $\gamma$, and $N$-(or nul) duals of $X$, respectively. It is known that $X \subset Y$, then $Y\eta \subset X^\eta$ for $\eta = \alpha$, $\beta$, $\gamma$, and $N$, and $c_0^N = l_\infty, l_\infty^N = c^N = c_0$ [2, 3].

**Lemma 3.2** (see [4]). Let $m$ be a positive integer. Then there exist positive constants $M_1$ and $M_2$ such that

$$M_1k^m \leq \frac{(m+k)}{k} \leq M_2k^m \quad \forall k = 0, 1, \ldots.$$  

(3.2)

**Lemma 3.3.** Let $x \in \Delta^m(c_0)$, then $(\frac{m+k}{k})^{-1} |x_k| \to 0, (k \to \infty)$.

**Proof.** The proof is trivial.  

\[\Box\]

**Theorem 3.4.** Let $m$ be a positive integer. Then $(\Delta^m(l_\infty))^N = (\Delta^m(c))^N = U_1$ and $(\Delta^m(c_0))^N = U_2$, where $U_1 = \{ a = (a_n) : (n^m a_n) \in c_0 \}$ and $U_2 = \{ a = (a_n) : \sum_{k=0}^{n} (n+m-k-1)a_n \} \in l_\infty$.

**Proof.** The proof of the part $(\Delta^m(l_\infty))^N = (\Delta^m(c))^N = U_1$ is easy. We show that $(\Delta^m(c_0))^N = U_2$. It is clear that $\sum_{k=0}^{n} (n+m-k-1) = (n+m) = (n+m)$. Let $a \in U_2$ and $x \in \Delta^m(c_0)$. Then

$$\lim_{n} a_n x_n = \lim_{n} \left( \sum_{k=0}^{n} \left( \begin{array}{c} n+m-k-1 \\ m-1 \end{array} \right) \right) a_n \left( \sum_{k=0}^{n} \left( \begin{array}{c} n+m-k-1 \\ m-1 \end{array} \right) \right)^{-1} x_n = 0.$$  

(3.3)

Hence $a \in (\Delta^m(c_0))^N$.
Now let \( a \in (\Delta^m(c_0))^N \). Then \( \lim_n a_n x_n = 0 \) for all \( x \in \Delta^m(c_0) \). On the other hand, for each \( x \in \Delta^m(c_0) \) there exists one and only one \( y = (y_k) \in c_0 \) such that

\[
x_n = \sum_{k=1}^n \binom{n + m - k - 1}{m - 1} y_k = \sum_{k=0}^n \binom{n + m - k - 1}{m - 1} y_k, \quad y_0 = 0,
\]

by (1.9). Hence

\[
\lim_n a_n x_n = \lim_n \sum_{k=0}^n \binom{n + m - k - 1}{m - 1} a_n y_k = 0 \quad \forall y \in c_0.
\]

If we take

\[
a_{nk} = \begin{cases} \binom{n + m - k - 1}{m - 1} a_n, & 1 \leq k \leq n, \\ 0, & k > n, \end{cases}
\]

then, we get

\[
\lim_n \sum_{k=0}^{\infty} a_{nk} y_k = \lim_n \sum_{k=0}^n \binom{n + m - k - 1}{m - 1} a_n y_k = 0 \quad \forall y \in c_0.
\]

Hence \( A \in (c_0, c_0) \) and so \( \sup_n \sum_{k=0}^{\infty} |a_{nk}| = \sup_n \sum_{k=0}^n (n + m - k - 1) |a_n| < \infty \). This completes the proof.

**Definition 3.5.** Let \( X \) be a sequence spaces, \( p > 0 \) and define

\[
X^{p\alpha} = \left\{ a = (a_k) : \sum_k |a_k x_k|^p < \infty, \quad \forall x \in X \right\},
\]

\[
X^{p\beta} = \left\{ a = (a_k) : \sum_k (a_k x_k)^p \text{ is convergent}, \quad \forall x \in X \right\},
\]

\[
X^{p\gamma} = \left\{ a = (a_k) : \sup_n \left| \sum_{k=0}^n (a_k x_k)^p \right| < \infty, \quad \forall x \in X \right\},
\]

then \( X^{p\alpha}, X^{p\beta}, X^{p\gamma} \) are called the \( p\alpha, p\beta, \) and \( p\gamma \)-duals of \( X \), respectively. It can be shown that \( X^{p\alpha} \subset X^{p\beta} \subset X^{p\gamma} \). If we take \( p = 1 \) in this definition, then we obtain the \( \alpha, \beta, \) and \( \gamma \)-duals of \( X \).

**Theorem 3.6.** Let \( X \) stand for \( l_{\infty}, c, \) and \( c_0 \) and \( 0 < p < \infty \). Then \( X^{p\eta} = U \), for \( \eta = \alpha, \beta, \) or \( \gamma \), where \( U = \{ a = (a_k) : \sum_k |a_k|^p < \infty \} = l_p \).

**Proof.** We give the proof for the case \( X = c_0 \) and \( \eta = \alpha \). If \( a \in U \), then

\[
\sum_k |a_k x_k|^p \leq \sup_k |x_k|^p \sum_k |a_k|^p < \infty
\]

for each \( x \in c_0 \). Hence \( a \in (c_0)^{p\alpha} \).
Now suppose that \( a \in (c_0)^{\rho_\alpha} \) and \( a \notin U \). Then there is a strictly increasing sequence \( (n_i) \) of positive integers \( n_i \) such that
\[
\sum_{k=n_i+1}^{k=n_{i+1}} |a_k|^p > ip.
\]
(3.10)

Define \( x \in c_0 \) by \( x_k = \frac{\text{sgn} a_k}{i} \) for \( n_i < k \leq n_{i+1} \) and \( x_k = 0 \) for \( 1 \leq k \leq n_1 \). Then we may write
\[
\sum_k |a_k x_k|^p = \sum_{k=n_1+1}^{k=n_2} |a_k x_k|^p + \cdots + \sum_{k=n_{i+1}}^{k=n_{i+1}} |a_k x_k|^p + \cdots
\]
\[
= \sum_{k=n_1+1}^{k=n_2} |a_k|^p + \cdots + \frac{1}{p^n} \sum_{k=n_{i+1}}^{k=n_{i+1}} |a_k|^p + \cdots
\]
\[
> 1 + 1 + \cdots = \sum_k 1 = \infty.
\]
(3.11)

This contradicts to \( a \in (c_0)^{\rho_\alpha} \). Hence \( a \in U \). The proof for the cases \( X = l_\infty \) or \( c \) and \( \eta = \beta \) or \( \gamma \) is similar.

The proofs of Lemmas 3.7 and 3.8 and Theorem 3.10 are easily obtained by using the same techniques of Mishra [5, Lemmas 1 and 2 and Theorem 1], therefore we give them without proofs.

**Lemma 3.7.** Let \( A : \Delta^m(l_\infty) \to E \) defines a matrix map. If \( A \) is weakly compact, then \( \sum_k a_k \) is unconditionally convergent in \( E \).

**Lemma 3.8.** If \( \sum_k a_k \) is unconditionally convergent in \( E \), then \( A : \Delta^m(l_\infty) \to E \) defines a matrix map, and \( A(\alpha) = \sum_k a_k \alpha_k \) for every \( \alpha = (\alpha_k) \in \Delta^m(l_\infty) \).

**Corollary 3.9.** If \( \sum_k a_k \) is unconditionally convergent in \( E \), then \( \Delta^m(l_\infty) \subseteq E_A \).

**Theorem 3.10.** If \( A : \Delta^m(l_\infty) \to E \) is a weakly compact matrix map, then \( A \) is compact map.

**Corollary 3.11.** Let \( E \) be a BK-space such that it contains no subspace isomorphic to \( \Delta^m(l_\infty) \). If \( A : \Delta^m(l_\infty) \to E \) defines a matrix map, then \( A \) is compact map.

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**References**


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