ON A SUBGROUP OF THE AFFINE WEYL GROUP \( \tilde{C}_4 \)

MUHAMMAD A. ALBAR

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Abstract. We study a subgroup of the affine Weyl group \( \tilde{C}_4 \) and show that this subgroup is a homomorphic image of the triangle group \( \triangle (3,4,4) \).

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1. Introduction. In the algebraic structures of the Coxeter groups \( \tilde{A}_{n-1}, B_n, D_n \), we observe the following. \( \tilde{A}_{n-1} \) is the subgroup of the wreath product \( Z_2 S_n \) such that \( \tilde{A}_{n-1} \cong D_3^\infty \rtimes S_3 \), where \( D_3^\infty \) is the subgroup of \( Z^n \) consisting of all elements of exponent sum zero [2]; \( D_n \) is a subgroup of \( B_n \cong Z_2 S_n \) such that \( D_n \cong Z_2^{n-1} \rtimes S_n \) and \( Z_2^{n-1} \) is the subgroup of \( Z_2^n \) containing all elements of exponent sum zero [4]. We have the following natural question about \( \tilde{C}_n \cong D_n^\infty \rtimes S_{n-1} \). What is the subgroup \( K \) of \( \tilde{C}_n \), where \( K \equiv H \rtimes S_{n-1} \) and \( H \) is the subgroup of \( D_n^\infty \) consisting of all elements of exponent sum zero [3]. In this paper we answer the question for \( n = 4 \) and find that the subgroup \( H \rtimes S_3 \) is a factor group of the triangle group \( \triangle (3,4,4) \).

We begin by giving a presentation for the direct product of three copies of the infinite dihedral group

\[
D_3^\infty = \langle a_1, a_2, a_3, b_1, b_2, b_3 \mid a_i^2 = b_i^2 = e, 1 \leq i \leq 3; \ a_i a_j = a_j a_i, 1 \leq i < j \leq 3; \ b_i b_j = b_j b_i, 1 \leq i < j \leq 3; \ a_i b_j = b_j a_i \text{ if } i \neq j, 1 \leq i, j \leq 3 \rangle.
\]

A presentation for the symmetric group of degree 3 is

\[
S_3 = \langle x_1, x_2 \mid x_1^2 = x_2^2 = (x_1 x_2)^3 = e \rangle.
\]

In [3], it is shown that \( \tilde{C}_4 \) is the semi-direct product \( \tilde{C}_4 \cong D_3^\infty \rtimes S_3 \) with the natural action

\[
(a_1, a_2, a_3)^{x_1} = (a_2, a_1, a_3), \ (a_1, a_2, a_3)^{x_2} = (a_1, a_3, a_2),
\]

\[
(b_1, b_2, b_3)^{x_1} = (b_2, b_1, b_3), \ (b_1, b_2, b_3)^{x_2} = (b_1, b_3, b_2).
\]

We consider the subgroup \( H \) of \( D_3^\infty \) containing all elements of exponent sum zero. \( H \) is a normal subgroup of \( D_3^\infty \) and \( D_3^\infty / H \cong \langle a_1 \mid a_1^2 = e \rangle \). Using the Reidemeister-Schreier
process we find the following presentation for $H$:

$$H = \langle y_1, y_2, y_3, y_4, y_5 \mid y_1^2 = y_2^2 = y_3^2 = (y_2 y_3)^2 = (y_3 y_4)^2 = (y_2 y_4)^2 \rangle,$$

where $y_1 = a_1 b_3, y_2 = a_2 a_1, y_3 = a_1 a_3, y_4 = a_1 b_1, y_5 = a_1 b_2$. From the action of $S_3$ on $D_3$ we easily compute the following action of $S_3$ on $H$:

$$\phi(y_1, y_2, y_3, y_4, y_5)^{x_1} = (y_2 y_1, y_2, y_2 y_3, y_2 y_5, y_2 y_4),$$

$$\phi(y_1, y_2, y_3, y_4, y_5)^{x_2} = (y_5, y_3, y_2, y_4, y_1).$$

2. **The group $H \rtimes S_3$.** We use the method of presentation of group extensions described in [1] to find a presentation for $H \rtimes S_3$ with the action computed in Section 1. A presentation for $H \rtimes S_3$ is

$$H \rtimes S_3 = \langle x_1, x_2, y_1, y_2, y_3, y_4, y_5 \mid RH, RS_3, H^S \rangle,$$

where $RH$ are the relations of $H, RS_3$ are the relations of $S_3$, the relations $H^S$ are the action of $S_3$ on $H$. Lengthy computations using Tietze transformations give the following presentation for $H \rtimes S_3$,

$$H \rtimes S_3 = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^4 = (ca)^4 = (bacac)^3 = e \rangle.$$

We observe that if $\triangle(3, 4, 4)$ is the hyperbolic triangle group generated by $a$, $b$, and $c$ and $N$ is the normal closure of $(bcac)^3$ in $\triangle(3, 4, 4)$, then $H \rtimes S_3$ is the factor group $(\triangle(3, 4, 4)) / N$.

3. **The triangle group $\triangle(3, 4, 4)$.** The triangle group $\triangle(3, 4, 4)$ is given by the presentations

$$\triangle(3, 4, 4) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^4 = (ca)^4 = e \rangle.$$

It is one of the hyperbolic triangle groups. $\triangle(3, 4, 4)$ is SQ-universal [6]. We find the derived subgroup of $\triangle(3, 4, 4)$ and show that it is SQ-universal using a method different from that in [7]. We also compute the growth series (word growth in the sense of Milnor and Gromov) of $\triangle(3, 4, 4)$. Using the Reidemeister-Schreier process we find that $\triangle'(3, 4, 4)$ is

$$\triangle'(3, 4, 4) = \langle x, y, z \mid x^2 = y^2 = (xy)^3 = (yz^{-1})^2 = e \rangle.$$

We consider the map $\theta : \triangle(3, 4, 4) \to Z_2 = \langle v \mid v^2 = e \rangle$ defined by $\theta(x) = \theta(y) = \theta(z) = v$. It is easy to see that

$$\ker \theta = \langle a, b, c, d \mid (ab)^2 = c^3 = d^3 = (ab^{-1})^2 = (bd^{-1})^2 = e \rangle.$$

We define another map $\phi : \ker \theta \to Z_2 = \langle u | u^2 = e \rangle$ by $\phi(a) = \phi(b) = u$ and $\phi(c) = \phi(d) = e$. Then $\ker \phi$ has the presentation

$$\ker \phi = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_3^2 = x_4^3 = x_5^3 = x_6^3 = (x_1 x_2)^2 \rangle,$$

$$= (x_1 x_4)^3 = x_2 x_6 x_3 x_5^{-1} = x_3 x_5^{-1} x_2 x_6^{-1} = e.$$
Letting $x_1 = x_5 = x_6 = e$ and $x_2 = x_3$ in $\ker \phi$ we get $(x_2, x_4 | x_2^2 = x_4^3 = e) = Z_2 \ast Z_3$. Since the free product $Z_2 \ast Z_3$ is $SQU$ [7], therefore $\ker \theta$ is $SQU$. But $\ker \theta$ is of finite index in $\Delta(3, 4, 4)$. Hence $\Delta(3, 4, 4)$ is $SQU$ [7]. The growth series of $\Delta(3, 4, 4)$ is computed using exercise 26 in Section 1 of Chapter 4 in Bourbaki [5] as

$$\gamma(t) = \frac{(1 + t)(1 + t + t^2)(1 + t + t^2 + t^3)}{1 - t^2 - 2t^3 - t^4 + t^6}. \quad (3.5)$$

We observe that zeros of the denominator of $\gamma(t)$ are not in the unit circle which implies that $\Delta(3, 4, 4)$ does not have a nilpotent subgroup of finite index. This is also known since $\Delta(3, 4, 4)$ is $SQU$.

**Remark 3.1.** It is interesting to know what subgroup of $\tilde{C}_n$ we get for $n > 4$. We did not find that yet.

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**References**


Muhammad A. Albar: Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia