ELEMENTS IN EXCHANGE RINGS WITH RELATED COMPARABILITY

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ABSTRACT. We show that if \( R \) is an exchange ring, then the following are equivalent: (1) \( R \) satisfies related comparability. (2) Given \( a, b, d \in R \) with \( aR + bR = dR \), there exists a related unit \( w \in R \) such that \( a + bt = dw \). (3) Given \( a, b \in R \) with \( aR = bR \), there exists a related unit \( w \in R \) such that \( a = bw \). Moreover, we investigate the dual problems for rings which are quasi-injective as right modules.

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Let \( R \) be an associative ring with identity. From [6], \( R \) is said to satisfy related comparability provided that for any idempotents \( e, f \in R \) with \( e = 1 + ab \) and \( f = 1 + ba \) for some \( a, b \in R \), there exists a \( u \in B(R) \) such that \( ueR \cong u fR \) and \( (1 - u)fR \cong (1 - u)eR \). The class of rings satisfying related comparability is quite large. It includes regular rings satisfying general comparability [10], one-sided unit regular rings [8] and partially unit-regular rings, while there still exist rings satisfying related comparability, which belong to none of the above classes (cf., [7, Example 10]).

In [4, 5], we studied related comparability over regular rings. In [6, 7], we investigated related comparability over exchange rings. It is shown that every exchange ring satisfying related comparability is separative [1]. Also, we show that related comparability over exchange rings is a Morita invariant. \( R \) is said to be an exchange ring if for every right \( R \)-module \( A \) and any two decompositions \( A = M \oplus N = \bigoplus_{i \in I} A_i \), where \( M_R \cong R \) and the index set \( I \) is finite, then there exist submodules \( A_i' \subseteq A_i \) such that \( A = M \oplus \bigoplus_{i \in I} A_i' \). Many authors have investigated exchange rings with some kind of comparability properties so as to study problems related partial cancellation properties of modules (see [1, 2, 6, 7, 12, 13]).

In this paper, we investigate related comparability over exchange rings by related units. Recall that \( w \in R \) is said to be a related unit of \( R \) if there exists some \( e \in B(R) \) such that \( w = eu + (1 - e)v \) for some \( u, v \in R \), where \( eu \) is right invertible in \( eR \) and \( (1 - e)v \) is left invertible in \( (1 - e)R \). \( w \in R \) is said to be a semi-related unit of \( R \) if \( w \in R \) is a related unit modulo \( J(R) \). By virtue of semi-related units, we also give some new element-wise properties of rings which are quasi-injective as right modules.

Throughout, all rings are associative with identities. \( B(R) \) denotes the set of all central idempotents of \( R \) and \( r \cdot \text{ann}(b)(1 \cdot \text{ann}(b)) \) denotes the right (left) annihilator of \( b \in R \).
**Lemma 1.** Let $R$ be an exchange ring. Then $R$ satisfies related comparability if and only if so does the opposite ring $R^{\text{op}}$ of $R$.

**Proof.** Since $R$ is an exchange ring, by virtue of [11, Proposition], so is the opposite ring $R^{\text{op}}$ of $R$. Assume that $R$ satisfies related comparability. Given $a^{\text{op}}, b^{\text{op}} \in R^{\text{op}}$ with $a^{\text{op}}x^{\text{op}} + b^{\text{op}} = 1^{\text{op}}$, then we have $xa + b = 1$ in $R$. In view of [6, Theorem 4], there exists a $y \in R$ such that $x + by$ is a related unit of $R$. Thus, we have some $e \in B(R)$ such that $(x + by)e$ is right invertible in $eR$ and $(x + by)(1 - e)$ is left invertible in $(1 - e)R$. By [5, Lemma 4], we claim that there are $z_1, z_2 \in R$ such that $(a + z_1b)e$ is left invertible in $eR$ and $(a + z_2b)(1 - e)$ is right invertible in $(1 - e)R$. Let $z = z_1e + z_2(1 - e)$. Then $a + zb$ is a related unit of $R$. Consequently, $a^{\text{op}} + b^{\text{op}}z^{\text{op}}$ is a related unit of $R^{\text{op}}$. By [6, Theorem 4], we conclude that $R^{\text{op}}$ satisfies related comparability. The converse is clear from $R \cong (R^{\text{op}})^{\text{op}}$. 

**Theorem 2.** Let $R$ be an exchange ring. Then the following are equivalent:

1. $R$ satisfies related comparability.
2. Given $a, b, d \in R$ with $aR + bR = dR$, there exists a related unit $w \in R$ such that $a + bt = dw$.
3. Given $a, b$ with $aR = bR$, there exists a related unit $w \in R$ such that $a = bw$.
4. Given $a, b, d \in R$ with $Ra + Rb = Rd$, there exists a related unit $w \in R$ such that $a + tb = wd$.
5. Given $a, b$ with $Ra = Rb$, there exists a related unit $w \in R$ such that $a = wb$.

**Proof.** (2)$\Rightarrow$(1). Trivial from [6, Theorem 4].

(1)$\Rightarrow$(2). Given $a, b, d \in R$ with $aR + bR = dR$. Let $g : dR \to dR/br$ be the canonical map, $f_1 : R \to aR$ given by $r \mapsto ar$ for any $r \in R$, $f_2 : R \to bR$ given by $r \mapsto br$ for any $r \in R$, $f_3 : R \to dR$ given by $r \mapsto dr$ for any $r \in R$. Since $aR + bR = dR$, we know that $gf_1, gf_2$ are epimorphisms. On the other hand, $R$ is a projective $R$-module. So there is some $\alpha \in \text{End}_R R$ such that $gf_1 = gf_2 = \alpha$. Since $gf_1$ is an epimorphism, we also have some $\psi \in \text{End}_R R$ such that $gf_3 = \alpha = \psi = f_3$. From $\alpha \psi + (1 - \alpha \psi) = 1$, there is a $\gamma \in \text{End}_R R$ such that $\alpha + (1 - \alpha \psi)\gamma = w$ is a related unit of $\text{End}_R R$. Therefore, we see that $gf_1 = gf_3\gamma = f_3(\alpha + (1 - \alpha \psi)\gamma) = f_3w$, and then $g(f_1 - f_3w) = 0$. Thus, we have $\text{Im}(f_1 - f_3w) \subseteq \ker g = bR$. By the projectivity of right $R$-module $R$, there exists some $\beta \in \text{End}_R R$ such that $f_2\beta = f_1 - f_3w$. Therefore, we claim that $a + b\beta(1) = f_1(1) + f_2(1)\beta(1) = f_3(1)w(1) = dw(1)$. It is easy to verify that $w(1)$ is a related unit of $R$.

(1)$\Rightarrow$(3). Given $a, b \in R$ with $aR = bR$, there exist $s, t \in R$ such that $a = bs$ and $b = at$. Thus, $b = bst$. Since $st + (1 - st) = 1$, by virtue of [6, Theorem 4], there exists some $z \in R$ such that $s + (1 - st)z = w$ is a related unit of $R$. Hence $a = bs = b(s + (1 - st)z) = bw$, as desired.

(3)$\Rightarrow$(1). Given any regular $a \in R$. Then there exists some $b \in R$ such that $a = aba$, so $aR = abR$. Thus $a = abw$ for some related unit $w \in R$. Since $ab + (1 - ab) = 1$, we see that $a + (1 - ab)w = (ab + (1 - ab))w = w$. By [5, Lemma 4], there is some $z \in R$ such that $b + z(1 - ab) = m$ is a related unit of $R$. Hence $a = aba = a(b + z(1 - ab))a = ama$. According to [6, Theorem 2], we claim that $R$ satisfies related comparability.

(1)$\Leftrightarrow$(4)$\Leftrightarrow$(5). By [11, Proposition], we see that the opposite ring $R^{\text{op}}$ of $R$ is
Thus, we can find some $k \in R$ such that $a + k$ is a related unit. We can also check that the range of $R$ behaves in a similar manner. This, however, is left to the reader.

**COROLLARY 3.** Let $R$ be an exchange ring. Then the following are equivalent:

1. $R$ satisfies related comparability.
2. Given $a, b \in R$ with $aR + r \cdot \text{ann}(b) = R$, there exists some $k \in R$ such that $a + k$ is a related unit.
3. Given $a, b \in R$ with $Ra + l \cdot \text{ann}(b) = R$, there exists some $k \in l \cdot \text{ann}(b)$ such that $a + k$ is a related unit.

**Proof.** (1) $\Rightarrow$ (2). Given $a, b \in R$ with $aR + r \cdot \text{ann}(b) = R$, then there exist $x \in R$, $k \in r \cdot \text{ann}(b)$ such that $ax + k = 1$. Since $R$ satisfies related comparability, by virtue of [6, Theorem 4], we can find a $\gamma \in R$ such that $a + ky$ is a related unit of $R$. It is easy to check that $ky \in r \cdot \text{ann}(b)$, as required.

(2) $\Rightarrow$ (1). Given $a, b \in R$ with $aR = bR$, there exist $s, t \in R$ such that $a = bs$ and $b = at$. Obviously, $1 - st \in r \cdot \text{ann}(b)$. Since $st + (1 - st) = 1$, we have $sR + r \cdot \text{ann}(b) = R$. Thus, we can find some $k \in r \cdot \text{ann}(b)$ such that $s + k = w$ is a related unit of $R$, and then $a = bs = b(s + k) = bw$, as asserted.

(1) $\equiv$ (3). Trivial by the symmetry of related comparability.

Recall that $n$ is in the stable range of $R$ provided that $a_1 R + \cdots + a_n R = R$ with $a_1, \ldots, a_n \in R$ implies that $(a_1 + a_n R_1)R + \cdots + (a_n + a_n R_n)R = R$ for some $b_1, \ldots, b_n \in R$. If no such $n$ exists, we say the stable range of $R$ is $\infty$, $x \in R$ is said to be related unit-regular if $x = xu x$ for some related unit $u \in R$. Now, we investigate related comparability by related unit-regularity as follows.

**PROPOSITION 4.** Let $R$ be an exchange ring with the finite stable range. Then the following are equivalent:

1. $R$ satisfies related comparability.
2. Given $a, b, d \in R$ with $aR + bR = dR$, there exist some related unit-regular $w_1, w_2 \in R$ such that $aw_1 + bw_2 = d$.
3. Given $a, b, d \in R$ with $Ra + Rb = Rd$, there exist some related unit-regular $w_1, w_2 \in R$ such that $w_1 a + w_2 b = d$.

**Proof.** (1) $\Rightarrow$ (2). Given $aR + bR = dR$ with $a, b, d \in R$. For right $R$-module $R^2$, the two sets $\{a, b\}$ and $\{0, d\}$ generate the same right $R$-submodule of $R^2$. Thus, we can find $A, B \in M_2(R)$ such that $(a, b) = (0, d)A$, $(0, d) = (a, b)B$. Assume that $A = (a_{ij})$, $B = (b_{ij})$, $I_2 - AB = (c_{ij}) \in M_2(R)$. Since $AB + (I_2 - AB) = I_2$, we have $(a_{21}, a_{22})(b_{12}, b_{22})^T + c_{22} = 1$. Since $R$ is an exchange ring satisfying related comparability, its stable range can only be $1, 2$ or $\infty$ by [7, Theorem 3]. So 2 is in the stable range of $R$. Thus, we have some $(y_1, y_2) \in R^2$ such that $(a_{21}, a_{22}) + c_{22}(y_1, y_2) \in R^2$ is unimodular. Set $Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$. Then, we claim that the second row of $A + (I_2 - AB)Y = U$ is unimodular. Clearly, $(0, d)U = (0, d)A = (a, b)$. Since $u_{21} R + u_{22} R = R$, we can find orthogonal idempotents $e_1 \in u_{21} R$, $e_2 \in u_{22} R$ such that $e_1 + e_2 = 1$. Assume that $e_1 = u_{21} x_1$, $e_2 = u_{22} x_2$. Let $w_1 = x_1 e_1$, $w_2 = x_2 e_2$. Then $w_1$ and $w_2$ are both regular in $R$. Moreover, we have $u_{21} w_1 + u_{22} w_2 = 1$. By the related comparability of $R$, we claim that both $w_i$ are related unit-regular, as asserted.
We have
\[ x = xyx \text{ for } y \in R. \]
So we have \( xR + (1 - xy)R = R \), and then \( xw_1 + (1 - xy)w_2 = 1 \) for some
related unit-regular \( w_1, w_2 \in R \).
We easily check that \( x + (1 - xy)w_2s \in R \) is related unit for some \( s \in R \).
Hence \( y + t(1 - xy) = w \), i.e., a related unit of \( R \).
Consequently, we show that \( x = xyx = xw_1x \), as desired.

(1)\( \iff \)(3). Clear from the symmetry of related comparability.

Recall that a module \( M \) is quasi-injective if any homomorphism of a submodule of \( M \) into \( M \) extends to an endomorphism of \( M \). Now, we investigate rings which are quasi-injective as right modules. These extend the corresponding results in [3].

**Lemma 5.** Let \( R \) be quasi-injective as a right \( R \)-module. Given \( a, b \in R \) with \( aR + bR = R \), there exists some \( t \in R \) such that \( a + bt \) is a semi-related unit.

**Proof.** Given \( a, b \in R \) with \( aR + bR = R \), then \( a(R/J(R)) + b(R/J(R)) = R/J(R) \).
Since \( R \) is quasi-injective as a right \( R \)-module, by virtue of [9, Theorem 1], \( R/J(R) \) is a regular, right self-injective ring. Hence \( R \) is an exchange ring satisfying related comparability. According to Theorem 2, we can find a \( y \in R \) such that \( a + by = \bar{w} \) is a related unit of \( R/J(R) \). Therefore \( a + by = w + r \) for some \( r \in J(R) \). Clearly, \( w + r \) is a semi-related unit of \( R \), as desired.

**Theorem 6.** Let \( R \) be quasi-injective as a right \( R \)-module. Then the following hold:

1. Given \( a, b \in R \) with \( r \cdot \text{ann}(a) = r \cdot \text{ann}(b) \), there exists a semi-related unit \( w \in R \) such that \( a = wb \).
2. Given \( a, b \in R \) with \( 1 \cdot \text{ann}(a) = 1 \cdot \text{ann}(b) \), there exists a semi-related unit \( w \in R \) such that \( a = bw \).

**Proof.** (1) Given \( a, b \in R \) with \( r \cdot \text{ann}(a) = r \cdot \text{ann}(b) \).
Since \( R \) is quasi-injective as a right \( R \)-module, by [3, Lemma 3.2], we have \( Ra = Rb \). Assume that \( a = sb, b = ta \) for some \( s, t \in R \). Then \( b = tsb \). Consequently, there exists some \( y \in R \) such that \( t + (1 - ts)y \) is a semi-related unit of \( R \) by Lemma 5. Using [5, Lemma 4], we have some \( z \in R \) such that \( s + z(1 - ts) = w \) is a semi-related unit of \( R \). Therefore, we claim that \( a = sb = (s + z(1 - ts))b = wb \), as desired.

(2) Given \( a, b \in R \) with \( 1 \cdot \text{ann}(a) = 1 \cdot \text{ann}(b) \). Similarly to the consideration above, we have \( aR = bR \). Assume that \( a = bs, b = at \) for some \( s, t \in R \). Then \( b = bst \). From \( st + (1 - st) = 1 \), we can find a \( y \in R \) such that \( s + (1 - st)y = w \) is a semi-related unit of \( R \). Therefore \( a = bs = b(s + (1 - st)y) = bw \), whence the result.

**Corollary 7.** Let \( R \) be quasi-injective as a left \( R \)-module. Then the following hold:

1. Given \( a, b \in R \) with \( r \cdot \text{ann}(a) = r \cdot \text{ann}(b) \), there exists a semi-related unit \( w \in R \) such that \( a = wb \).
2. Given \( a, b \in R \) with \( 1 \cdot \text{ann}(a) = 1 \cdot \text{ann}(b) \), there exists a semi-related unit \( w \in R \) such that \( a = bw \).

**Proof.** Applying Theorem 6 to the opposite ring \( R^{\text{op}} \) of \( R \), we complete the proof.

**Theorem 8.** Let \( R \) be a ring which is quasi-injective as a right \( R \)-module. Then the following hold:
(1) Given $a, b \in R$ with $r \cdot \text{ann}(a) \cap r \cdot \text{ann}(b) = 0$, there exists $t \in R$ such that $a + tb$ is a semi-related unit.

(2) Given $a, b \in R$ with $1 \cdot \text{ann}(a) \cap 1 \cdot \text{ann}(b) = 0$, there exists $t \in R$ such that $a + bt$ is a semi-related unit.

**Proof.** (1) Given $a, b \in R$ with $r \cdot \text{ann}(a) \cap r \cdot \text{ann}(b) = 0$, by virtue of [3, Proposition 3.4], we know that $Ra + Rb = R$. Thus, $(R/J(R)) a + (R/J(R)) b = R/J(R)$. Since $R$ is a quasi-injective ring, from [9, Theorem 1], $R/J(R)$ is a regular, right self-injective ring. Moreover, we see that $R/J(R)$ satisfies related comparability. In view of Theorem 2, there exists $t \in R$ such that $a + t\bar{b} = \bar{w}$ with $w$ is a semi-related unit of $R$. Thus, there is some $k \in J(R)$ such that $a + tb = w + k$. Clearly, $w + k$ is also a semi-related unit. Thus, we claim that $a + tb$ is a semi-related unit of $R$.

(2) Given $a, b \in R$ with $1 \cdot \text{ann}(a) \cap 1 \cdot \text{ann}(b) = 0$, analogously to [3, Proposition 3.4], we claim that $aR + bR = R$. Thus $a(R/J(R)) + \bar{b}(R/J(R)) = R/J(R)$. Similarly to the consideration above, we show that $R/J(R)$ satisfies related comparability. In view of Theorem 2, there exists $t \in R$ such that $a + bt = w + k$ with $w$ is a semi-related unit and $k \in J(R)$. Since $w + k$ is also a semi-related unit, the result follows.

**Corollary 9.** Let $R$ be a ring which is quasi-injective as a left $R$-module. Then the following hold:

(1) Given $a, b \in R$ with $r \cdot \text{ann}(a) \cap r \cdot \text{ann}(b) = 0$, there exists $t \in R$ such that $a + tb$ is a semi-related unit.

(2) Given $a, b \in R$ with $1 \cdot \text{ann}(a) \cap 1 \cdot \text{ann}(b) = 0$, there exists $t \in R$ such that $a + bt$ is a semi-related unit.

**Proof.** Applying Theorem 8 to the opposite ring $R^{\text{op}}$ of $R$, we easily obtain the result.

Since every regular, right (left) self-injective ring is a quasi-injective ring with trivial Jacobson. As an immediate consequence of Theorem 6, Corollary 7, Theorem 8, and Corollary 9, we now derive the following.

**Corollary 10.** Let $R$ be a regular, right (left) self-injective ring. Then the following hold:

(1) Given $a, b \in R$ with $r \cdot \text{ann}(a) = r \cdot \text{ann}(b)$, there exists a related unit $w \in R$ such that $a = wb$.

(2) Given $a, b \in R$ with $1 \cdot \text{ann}(a) = 1 \cdot \text{ann}(b)$, there exists a related unit $w \in R$ such that $a = bw$.

(3) Given $a, b \in R$ with $r \cdot \text{ann}(a) \cap r \cdot \text{ann}(b) = 0$, there exists $t \in R$ such that $a + tb$ is a related unit.

(4) Given $a, b \in R$ with $1 \cdot \text{ann}(a) \cap 1 \cdot \text{ann}(b) = 0$, there exists $t \in R$ such that $a + bt$ is a related unit.

**References**


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