NEW CHARACTERIZATIONS OF SOME $L^p$-SPACES

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(Received 5 October 1998)

ABSTRACT. For a complete measure space $(X, \Sigma, \mu)$, we give conditions which force $L^p(X, \mu)$, for $1 \leq p < \infty$, to be isometrically isomorphic to $\ell^p(\Gamma)$ for some index set $\Gamma$ which depends only on $(X, \mu)$. Also, we give some new characterizations which yield the inclusion $L^p(X, \mu) \subset L^q(X, \mu)$ for $0 < p < q$.

Keywords and phrases. Complete measure space, $L^p$-spaces.

2000 Mathematics Subject Classification. Primary 28A20, 46E30.

1. Introduction. Suppose $X$ is a nonempty set, $\Sigma$ is $\sigma$-algebra of subsets of $X$, $\mu$ a positive measure on $\Sigma$. For each positive number $p$, let $L^p(X, \mu)$ denote the space of all real valued $\Sigma$-measurable functions $f$ on $X$ such that $\int_X |f|^p \, d\mu < \infty$, and $L^\infty(X, \mu)$ denote the space of all essentially bounded, real valued $\Sigma$-measurable functions on $X$. In [2, 3, 5] some characterizations of the positive measure $\mu$ on $(X, \Sigma)$ for which $L^p(X, \mu) \subset L^q(X, \mu)$, $0 < p < q$, are given. The purpose of this note is to give some new characterizations of such measure $\mu$ which yield the inclusion $L^p(X, \mu) \subset L^q(X, \mu)$ for $0 < p < q$. Our proofs are more transparent, direct, and work even if the measure $\mu$ is not $\sigma$-finite. Further we show that in a situation when $L^p(X, \mu) \subset L^q(X, \mu)$ for some pair $p, q$ with $0 < p < q$, then $L^p(X, \mu)$, for $1 \leq p < \infty$, is isometrically isomorphic to $\ell^p(\Gamma)$ for some index set $\Gamma$ which depends only on the measure space $(X, \Sigma, \mu)$.

2. Preliminaries. Throughout the following $(X, \Sigma, \mu)$ is a positive measure space. We assume that the measure $\mu$ is complete. For the sake of simplicity, we write $L^p(\mu)$ for $L^p(X, \mu)$ and $L^\infty(\mu)$ for $L^\infty(X, \mu)$. A set $A \in \Sigma$ is called an atom if $\mu(A) > 0$ and for every $E \subset A$ with $E \in \Sigma$, either $\mu(E) = 0$ or $\mu(E) = \mu(A)$. A measurable subset $E$ with $\mu(E) > 0$ is nonatomic if it does not contain any atom. We say that two atoms $A_1$ and $A_2$ are distinct if $\mu(A_1 \cap A_2) = 0$. We say that two atoms $A_1$ and $A_2$ are indistinguishable if $\mu(A_1 \cap A_2) = \mu(A_1) = \mu(A_2)$. A measurable space $(X, \Sigma, \mu)$ is said to be atomic if every measurable set of positive measure contains an atom. For more information on measurable spaces and related topics we refer to [1, 2, 4]. We collect some interesting and useful properties of atomic and nonatomic sets in the following proposition.

**Proposition 2.1.** Let $(X, \Sigma, \mu)$ be a complete measure space.

(a) If $\{A_n\}$ is a sequence of distinct atoms, then there exists a sequence $\{B_n\}$ of disjoint atoms such that for each $n$, $B_n \subseteq A_n$ and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.

(b) If $\{A_n\}$ is a sequence of distinct atoms, and $A$ is an atom contained in $\bigcup A_n$, then there exists a unique $m$ such that $A$ is indistinguishable from $A_m$. 

(c) If \( A \) is a nonatomic set of positive measure, then there exists a sequence \( \{E_n\} \) of disjoint measurable subsets of \( A \) of positive measure such that \( \mu(E_n) \to 0 \) as \( n \to \infty \).

(d) If \( f \in L^p(\mu) \) and \( A \) is an atom in \( \Sigma \), then \( f \) is constant almost everywhere (a.e.) on \( A \).

**Proof.** (a) Let \( B_1 = A_1 \) and \( B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k \). Obviously, \( B_i \)'s are disjoint and \( \cup A_n = \cup B_n \). Also, \( \mu(B_n) = \mu(A_n \setminus \bigcup_{k=1}^{n-1} A_k) \) is either zero or is equal to \( \mu(A_n) \). If \( \mu(B_n) = 0 \), then \( \mu(A_n) = \mu(A_n \cap (\bigcup_{k=1}^{n-1} A_k)) \leq \sum_{k=1}^{n-1} \mu(A_n \cap A_k) \). Since \( A_k \)'s are distinct atoms, this implies \( \mu(A_n) = 0 \), which is absurd. Hence \( \mu(B_n) = \mu(A_n) \).

(b) Suppose \( A \) is contained in \( \cup A_n \). From part (a) of the proposition, there exists a sequence \( \{B_n\} \) of disjoint atoms such that \( B_n \subseteq A_n \) for each \( n \) and \( \cup A_n = \cup B_n \).

Clearly \( \mu(A \cap B_n) \) is either zero or \( \mu(A) \) for each \( n \). Hence by (2.1), there exists a unique \( m \) such that \( \mu(A \cap B_m) = \mu(A) \). Since \( A \) and \( B_m \) are indistinguishable, \( B_m \subset A_m \), it follows that \( A \) and \( A_m \) are indistinguishable.

(c) Suppose \( A \) is a nonatomic set of positive measure and \( \mu(A) = \delta \). There exists a measurable subset \( E_1 \) of \( A \) such that \( 0 < \mu(E_1) < \delta/2 \). Since \( A \setminus E_1 \) is nonatomic, there exists a measurable subset \( E_2 \) of \( A \setminus E_1 \) such that \( 0 < \mu(E_2) < \delta/4 \). Having chosen \( E_1, E_2, \ldots, E_{n-1} \), choose a measurable subset \( E_n \) of \( A \setminus (E_1 \cup E_2 \cup \ldots \cup E_{n-1}) \) such that \( \mu(E_n) < \sigma/2^n \). Obviously, \( E_n \)'s are disjoint and \( \mu(E_n) \to 0 \) as \( n \to \infty \).

(d) Since \( A \) is an atom, it is enough to show that if \( f \) is integrable then \( f \) is constant a.e. on \( A \). Choose a real number \( c \) such that \( c \mu(A) = \int_A f(x) \, d\mu \). Let \( B = \{x \in A \mid f(x) \neq c\} \). We claim \( \mu(B) = 0 \). Obviously \( B = \{x \in A \mid f(x) < c\} \cup \{x \in A \mid f(x) > c\} \).

First, we show that \( \mu(\{x \in A \mid f(x) > c\}) = 0 \). We can use a similar argument to show that \( \mu(\{x \in A \mid f(x) > c\}) = 0 \). We note that \( \{x \in A \mid f(x) > c\} = \bigcup_{i=1}^{\infty} B_i \cup B_0 \), where \( B_i = \{x \in A \mid c + 1/(1 + i) \leq f(x) < c + 1/(1 + i)\} \) and \( B_0 = \{x \in A \mid f(x) \geq c + 1\} \). Obviously all \( B_i \)'s are disjoint. Since \( A \) is an atom, at most one of the \( B_i \)'s can have a positive measure. If \( B_k \) is positive for some \( k \), \( 0 \leq k < \infty \), then \( c \mu(A) = \int_A f(x) \, d\mu(x) = \int_{B_k} f(x) \, dx \geq (c + 1/(k + 1)) \mu(A) \). This is absurd. Therefore, \( \mu(B_k) = 0 \) for all \( i \geq 0 \). Hence \( \{x \in A \mid f(x) > c\} \) is of measure zero. This completes the proof.

The following lemmas are quite useful in the proof of the main result.

**Lemma 2.2.** Let \( (X, \Sigma, \mu) \) be a complete measure space.

(a) If \( \{B_n\} \) is a sequence of measurable sets of positive measure and \( \mu(B_n) \to 0 \) as \( n \to \infty \), then there exists a sequence \( \{C_n\} \) of disjoint measurable sets of positive measure such that \( \mu(C_n) \to 0 \) as \( n \to \infty \).

(b) If \( \{E_n\} \) is a sequence of disjoint measurable sets of positive measure such that \( \mu(E_n) \to 0 \) as \( n \to \infty \), then for any positive number \( m > 1 \) there exists a subsequence \( \{E_{n_k}\} \) of \( \{E_n\} \) and an increasing sequence \( \{k_i\} \) of positive integers such that \( \mu(E_{n_k}) \in ((1/k_i)^m, (1/k_i)^{m-1}] \).

**Proof.** (a) Without loss of generality, we may assume that \( \mu(B_n) < 1 \) for each \( n \). If for some positive integer \( k \), \( B_k \) is nonatomic, by using an argument similar to
that of Proposition 2.1(c), we can construct a sequence \( C_n \) of disjoint measurable sets of positive measure such that \( \mu(C_n) \to 0 \) as \( n \to \infty \). Suppose that \( B_k \) is atomic for each positive integer \( k \), let \( A_1 \) be an atom contained in \( B_1 \). Since \( \mu(B_n) \to 0 \) as \( n \to \infty \), \( \mu(A_1 \cap B_k) \) can be positive only for finitely many \( k > 1 \). Let \( n_1 \) be the smallest positive integer such that \( \mu(A_1 \cap B_{n_1}) = 0 \). Now choose an atom \( A_2 \) contained in \( B_{n_1} \). Obviously \( A_2 \) is indistinguishable from \( A_1 \). Also, \( \mu(A_2 \cap B_k) \) can be positive for at most finitely many \( k \) greater than \( n_1 \). Let \( n_2 \) be the smallest positive integer greater than \( n_1 \) such that \( \mu(A_2 \cap B_{n_2}) = 0 \). Now choose an atom \( A_3 \) contained in \( B_{n_2} \). Clearly \( A_3 \) is indistinguishable from \( A_1 \) and \( A_2 \). Continuing in this fashion, we get a sequence \( \{A_k\} \) of atoms which are indistinguishable and \( A_k \subseteq B_{n_{k-1}} \) for each \( k \geq 2 \). By Proposition 2.1(a), we may choose a sequence \( \{E_k\} \) of disjoint atoms such that \( E_k \subseteq A_k \). Clearly, \( 0 < \mu(E_k) = \mu(A_k) \leq \mu(B_{n_{k-1}}) \). This completes the proof of part (a).

(b) Let \( \{E_n\} \) be a sequence of measurable sets of positive measure such that \( \mu(E_n) \to 0 \) as \( n \to \infty \). Without loss of generality, we may assume that \( \{\mu(E_n)\} \) is a strictly decreasing sequence. Let \( m > 1 \). Let \( k_0 > 2 \) be a positive integer such that \( 1/2 < \left( k/(k+1) \right)^{m-1} \) for all \( k \geq k_0 \). Clearly \( \left( (1/\ell + 1)^m, 1/(\ell + 1)^{m-1} \right) \cap \left( (1/\ell)^m, (1/\ell)^{m-1} \right) \) is nonempty for each \( \ell \geq k_0 \). Since \( \mu(E_n) \) is decreasing to zero, the set \( \{\mu(E_n) \mid n \geq 1\} \) must have a nonempty intersection with an interval \( ((1/k)^m, (1/k)^{m-1}) \) for some \( k \geq k_0 \). Let \( k_1 \) be the smallest positive integer greater than \( k_0 \) such that \( \{\mu(E_n) \mid n \geq 1\} \cap ((1/k_1)^m, (1/k_1)^{m-1}) \neq \emptyset \). Let \( n_1 \) be the smallest positive integer such that \( \mu(E_{n_1}) \in ((1/k_1)^m, (1/k_1)^{m-1}) \). Next choose the smallest integer \( k_2 \) greater than \( k_1 \) such that \( \{\mu(E_n) \mid n > n_1\} \cap ((1/k_2)^m, (1/k_2)^{m-1}) \neq \emptyset \). Let \( n_2 \) be the smallest integer greater than \( n_1 \) such that \( \mu(E_{n_2}) \in ((1/k_2)^m, (1/k_2)^{m-1}) \). Continuing inductively in this way, we can choose strictly increasing sequences of positive integers \( \{k_i\} \) and \( \{n_i\} \) such that \( \mu(E_{n_i}) \in ((1/k_i)^m, (1/k_i)^{m-1}) \). This completes the proof of part (b).

\[ \square \]

**Lemma 2.3.** If \( L^p(\mu) \subseteq L^q(\mu) \) for \( 0 < p < q \), then there does not exist a disjoint sequence \( \{E_n\} \) of measurable sets of positive measure such that \( \mu(E_n) \to 0 \) as \( n \to \infty \).

**Proof.** Suppose there exists a disjoint sequence \( \{E_n\} \) of measurable sets of positive measure such that \( \mu(E_n) \to 0 \) as \( n \to \infty \). Let

\[ m = 3 - \frac{3p}{p-q} = -\frac{3q}{p-q}. \quad (2.2) \]

Clearly \( m > 1 \). By Lemma 2.2(b), there exists a subsequence \( \{E_{n_i}\} \) of \( \{E_n\} \) and a strictly increasing sequence of positive integers \( \{k_i\} \) such that \( \mu(E_{n_i}) \in ((1/k_i)^m, (1/k_i)^{m-1}) \). Define a function \( f \) from \( X \) into real numbers by \( f(x) = (1/k_i)^{3p/(p-q)} \) if \( x \in E_{n_i} \) and \( f(x) = 0 \) for all \( x \notin \bigcup_{i=1}^{\infty} E_{n_i} \). Then

\[ \int_X |f(x)|^p d\mu = \sum_{i=1}^{\infty} \int_{E_{n_i}} |f(x)|^p d\mu = \sum_{i=1}^{\infty} \left( \frac{1}{k_i} \right)^{3p/(p-q)} \mu(E_{n_i}) \leq \sum_{i=1}^{\infty} \left( \frac{1}{k_i} \right)^{3p/(p-q)} \left( \frac{1}{k_i} \right)^{m-1} = \sum_{i=1}^{\infty} \left( \frac{1}{k_i} \right)^2 < \infty. \quad (2.3) \]
On the other hand,
\[
\int_X |f(x)|^q \, d\mu = \sum_{i=1}^{\infty} \int_{E_{n_i}} |f(x)|^q \, d\mu = \sum_{i=1}^{\infty} \left( \frac{1}{k_i} \right)^{3q/(p-q)} \mu(E_{n_i}) \\
\geq \sum_{i=1}^{\infty} \left( \frac{1}{k_i} \right)^{3q/(p-q)} \left( \frac{1}{k_i} \right)^m = \infty.
\]  
(2.4)

Thus \( f \in L^p(\mu) \) but \( f \notin L^q(\mu) \). This completes the proof of the lemma. \( \square \)

3. Main results. For the sake of clarity, we first start with a definition. For any nonempty set \( \Gamma \), and \( p > 0 \), we define \( \ell^p(\Gamma) \) to be the set of all extended real valued functions \( f \) on \( \Gamma \) such that \( f \) is nonzero only on a countable subset of \( \Gamma \) and \( \sum_{\alpha} |f(\alpha)|^p < \infty \).

When \( p \geq 1 \), \( \ell^p(\Gamma) \) becomes a Banach space under the norm defined by \( \| f \|_{\ell^p(\Gamma)} = (\sum_{\alpha} |f(\alpha)|^p)^{1/p} \). Now, we are ready to state the main result.

**Theorem 3.1.** Let \((X, \Sigma, \mu)\) be a complete measure space. The following six conditions are equivalent:

1. \( L^p(\mu) \subset L^q(\mu) \) for some pair of real numbers \( p \) and \( q \) with \( 0 < p < q \).
2. \( L^p(\mu) \subset L^\infty(\mu) \) for some \( p > 0 \).
3. \( L^p(\mu) \subset L^\infty(\mu) \) for all positive numbers \( p \).
4. \( L^p(\mu) \subset L^q(\mu) \) for all \( p \) and \( q \) with \( 0 < p < q \).
5. There is no sequence \( \{B_n\} \) in \( \Sigma \) such that \( \mu(B_n) > 0 \) for each \( n \) and \( \mu(B_n) \to 0 \) as \( n \to \infty \).
6. \((X, \Sigma, \mu)\) is atomic with \( \inf_{A \in \Pi} \mu(A) > 0 \), where \( \Pi \) is the set of all atoms in \( \Sigma \).

Moreover, these statements imply that: for each positive number \( p \geq 1 \), \( L^p(\mu) \) is isometrically isomorphic to \( \ell^p(\Gamma) \) for some index set \( \Gamma \) which depends only on \((X, \Sigma, \mu)\).

**Proof.** Since the implication (4)\( \Rightarrow \) (1) is obvious, in order to prove the equivalence of the statements (1) through (6), it is enough to prove the following implications:

1. \( \Rightarrow \) (2), (2)\( \Rightarrow \) (3), (3)\( \Rightarrow \) (4), (4)\( \Rightarrow \) (5), (5)\( \Rightarrow \) (6), and (6)\( \Rightarrow \) (2).

(1)\( \Rightarrow \) (2): suppose that \( L^p \subset L^q \) for some pair \( p, q \) with \( 0 < p < q \). We claim \( L^p \subset L^\infty \).

Suppose there is an \( f \) in \( L^p \) which is not essentially bounded. Then there exists a strictly increasing sequence \( \{n_k\} \) of positive integers such that for each \( k \geq 1 \), the set \( E_k = \{x \in X | n_k \leq |f(x)| < n_k + 1\} \) is of a positive measure. Obviously \( E_k \)'s are disjoint. Since \( \mu(E_k) n_k^p \leq \int_X |f|^p \, d\mu \leq \int_X |f|^p \, d\mu \), it follows \( \mu(E_k) \to 0 \). This is a contradiction in view of Lemma 2.2.

(2)\( \Rightarrow \) (3): suppose that \( L^p(\mu) \subset L^\infty(\mu) \) for some \( p > 0 \). Let \( r \) be any positive real number. We show \( L^r(\mu) \subset L^\infty(\mu) \). Let \( f \in L^r(\mu) \). If \( A = \{x : |f(x)| > 1\} \) is of measure zero, then obviously \( f \in L^\infty(\mu) \). Suppose that \( A \) is a positive measure. Let \( g = X_A f \), where \( X_A \) is the characteristic function of the set \( A \). Clearly, \( g \in L^r(\mu) \) and \( |g| \geq 1 \) a.e. Since \( |g|^r/p \in L^p, |g|^r/p \in L^\infty \) Let \( M = \text{ess sup} |g|^r/p \). Let \( \epsilon > 0 \). Choose \( \delta > 0 \) such that \( (M + \delta)^p/r - M^p/r < \epsilon \). Since \( \{x : |g(x)| > (M + \delta)^p/r\} \subset \{x : |g(x)| > (M + \delta)^p/r\} \), and \( \mu(\{x : |g(x)|^r/p > M + \delta\}) = 0 \), it follows that \( \text{ess sup} |g| \leq M^p/r \).

(3)\( \Rightarrow \) (4): suppose that \( L^p \subset L^\infty \) for all \( p \geq 0 \). Let \( g \in L^p \). Write \( A = \{x : |g(x)| > 1\} \). If \( A \) is a nonatomic set of positive measure, by Proposition 2.1(c), \( A \) contains a disjoint
sequence \( \{ E_n \} \) of measurable subsets of \( A \) of positive measure such that \( \mu(E_n) \to 0 \) as \( n \to \infty \). As is noted in the proof of Lemma 2.3, we can construct a function \( f \) in \( L^p \) which is not in \( L^\infty \). Hence \( A \) contains an atom. Since the measure of \( A \) is finite, in view of Proposition 2.1(a), \( A \) cannot contain infinitely many atoms. Therefore, \( A \) can be written as a finite disjoint union of atoms. Suppose that \( A = \bigcup_{i=1}^{\infty} \theta_i \), where \( \theta_i \)'s are disjoint atoms. By Proposition 2.1(d), \( g \) is constant on each \( \theta_i \). Let \( g_{\theta_i} \) be the value of \( g \) on \( \theta_i \). Then for any \( q > p \),

\[
\int_X |g|^q \, d\mu = \int_{X-A} |g|^q \, d\mu + \int_{A} |g|^q \, d\mu \\
\leq \int_{X-A} |g|^p \, d\mu + \sum_{i=1}^{n} |g_{\theta_i}|^q \mu(\theta_i) \\
\leq \int_{X} |g|^p \, d\mu + \sum_{i=1}^{n} |g_{\theta_i}|^q \mu(\theta_i) < \infty.
\]

Hence \( L^p \subset L^q \) for \( q > p \).

(4)\( \implies \) (5): this follows from Lemmas 2.2(a) and 2.3.

(5)\( \implies \) (6): Proposition 2.1(c) implies that the space \( (X,\Sigma,\mu) \) is atomic. Since atoms are of positive measure, obviously statement (5) implies that \( \inf_{A \in \pi} \mu(A) > 0 \).

(6)\( \implies \) (2): Suppose \( (X,\Sigma,\mu) \) is atomic with \( \inf_{A \in \pi} \mu(A) > 0 \). Let \( p > 0 \) and \( g \in L^p(\mu) \). Suppose \( \mu(B) > 0 \). Obviously \( \mu(B) \) is finite. Since \( \inf_{A \in \pi} \mu(A) > 0 \), \( B \) cannot contain infinitely many atoms. Therefore, \( B \) can be written as finite disjoint union of atoms. Since \( g \) is constant on each atom, it follows that \( g \in L^\infty \).

Finally, we show that for \( p \geq 1 \), one of the statements (1) through (6) (and hence all of them) imply statement (7). Let \( (X,\Sigma,\mu) \) be a measure space such that \( L^p(\mu) \subset L^q(\mu) \) for some \( 1 \leq p < q \). Let \( \{ \theta_i \}_{i \in \Gamma} \) be the collection of all atoms in \( X \) where \( \Gamma \) is some index set. Let \( f \in L^p(\mu) \) be an arbitrary nonzero element of \( f \). By Proposition 2.1(d) \( f \) is constant almost everywhere on any atom. We denote the value of \( f \) on an atom \( \theta \) lies in the support of \( f \) by \( f_{\theta} \). Since the support of \( f \) is \( \sigma \)-finite, and by statement (5) of the theorem any measurable set of finite measure is disjoint union of finitely many atoms, the support of \( f \) can be written as countable union of atoms. Let \( \{ \theta_n(f) \} \) be the set of all atoms that forms the support of \( f \). We define \( F : L^p(\mu) \rightarrow \ell^p(\Gamma) \) by

\[
F(f)(\gamma) = \begin{cases} 
 f_{\theta_n}(\mu(\theta_n))^{1/p}, & \text{if } \theta_y = \theta_n(f) \text{ for some } n, \\
 0, & \text{if } \theta_y \notin \{ \theta_n(f) \}
\end{cases}
\]

for any nonzero \( f \) in \( L^p(\mu) \). The function \( F \) is well defined since any two functions that are equal in \( L^p(\mu) \) are equal almost everywhere and thus share the same support. It is straightforward to verify that \( F \) is a one-to-one linear operator from \( L^p(\mu) \) into \( \ell^p(\Gamma) \). Let \( h \in \ell^p(\Gamma) \). Since \( h \) is nonzero only on a countable subset \( \Gamma_h \) of \( \Gamma \), define \( f \) on \( X \) as follows:

\[
f(x) = \begin{cases} 
 h(y), & \text{if } x \in \theta_y, \ y \in \Gamma_h, \\
 (\mu(\theta_y))^{1/p}, & \text{if } x \notin \bigcup_{y \in \Gamma_h} \theta_y.
\end{cases}
\]
Obviously, \( f \in L^p(\mu) \) and \( F(f) = h \). Thus \( F \) is an isomorphism from \( L^p(\mu) \) onto \( \ell^p(\Gamma) \). Further for any \( f \in L^p(\mu) \),

\[
\|F(f)\|_{\ell^p(\Gamma)}^p = \sum_i |f_{\theta_i}(\mu(\theta_i))^{1/p}|^p = \sum_i |f_{\theta_i}|^p \mu(\theta_i)
\]

\[
= \sum_i \int_{\theta_i} |f(x)|^p d\mu = \int_X |f(x)|^p d\mu = \|f\|^p,
\]

where the sum runs over \( i \in \Gamma \) such that \( \theta_i \) is in the support of \( f \).

Therefore \( F \) is an isometry. This completes the proof of the theorem. \( \square \)

**REFERENCES**


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