A GENERAL EXISTENCE PRINCIPLE FOR FIXED POINT THEOREMS IN D-METRIC SPACES

B. C. DHAGE, A. M. PATHAN, and B. E. RHOADES

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Abstract. We establish two general principles for fixed point theorems in D-metric spaces, and then show that several theorems in D-metric spaces follow as corollaries of these general principles.

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1. Introduction. The concept of a D-metric space was introduced by the first author in [1]. A nonempty set $X$, together with a function $D : X \times X \times X \to [0, \infty)$ is called a D-metric space, denoted by $(X, D)$ if $D$ satisfies

(i) $D(x, y, z) = 0$ if and only if $x = y = z$ (coincidence),
(ii) $D(x, y, z) = D(p\{x, y, z\})$, where $p$ is a permutation of $x, y, z$ (symmetry),
(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$ (tetrahedral inequality).

The nonnegative real function $D$ is called a D-metric on $X$. Some specific examples of D-metrics appear in [2]. A D-metric is a continuous function on $X^3$ in the topology of D-metric convergence, which is Hausdorff (see [5]).

In this paper, we establish two general fixed point principles for mappings in a D-metric space, which yield several fixed point theorems as corollaries.

2. Preliminaries. Let $f : X \to X$. The orbit of $f$ at the point $x \in X$ is the set $O(x) = \{x, f x, f^2 x, \ldots\}$. An orbit of $x$ is said to be bounded if there exists a constant $K > 0$ such that $D(u, v, w) \leq K$ for all $u, v, w \in O(x)$. The constant $K$ is called a D-bound of $O(x)$. A D-metric space $X$ is said to be $f$-orbitally bounded if $O(x)$ is bounded for each $x \in X$. A sequence $x_n \subset X$ is said to be D-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that, for all $m > n, p \geq n_0, D(x_m, x_n, x_p) < \varepsilon$. A sequence $\{x_n\} \subset X$ is said to be D-convergent to a point $x \in X$ if, for each $\varepsilon > 0$ there exists a positive integer $n_0$ such that, for all $m, n \geq n_0, D(x_m, x_n, x) < \varepsilon$. An orbit $O(x)$ is called $f$-orbitally complete if every D-Cauchy sequence in $O(x)$ converges to a point in $X$.

Lemma 2.1 [4]. Let $\{x_n\} \subset X$ be a bounded sequence with D-bound $K$ satisfying

$$D(x_n, x_{n+1}, x_m) \leq \lambda^n K \quad (2.1)$$

for all positive integers $m > n$, and some $0 \leq \lambda < 1$. Then $\{x_n\}$ is D-Cauchy.
3. Main results

**Theorem 3.1.** Let \((X, D)\) be a \(D\)-metric spaces, \(f\) a selfmap of \(X\). Suppose that there exists an \(x_0 \in X\) such that \(O(x_0)\) is \(D\)-bounded and \(f\)-orbitally complete. Suppose also that \(f\) satisfies

\[
D(fx, fy, fz) \leq \lambda \max \{D(x, y, z), D(x, fx, z)\}
\]

for some \(0 \leq \lambda < 1\). Then \(f\) has a unique fixed point in \(X\).

**Proof.** Suppose there exists an \(n\) such that \(x_n = x_{n+1}\). Then \(f\) has \(x_n\) as a fixed point in \(X\). Therefore we may assume that all of the \(x_n\) are distinct.

We wish to show that, for any positive integers \(m, n\), \(m > n\), that \(D(x_{n+1}, x_{n+2}, x_m) \leq \lambda^n K\), where \(K\) is the \(D\)-bound of \(O(x_0)\). The proof is by induction. From (3.1), for any \(m\),

\[
D(x_1, x_2, x_m) \leq \lambda \max \{D(x_0, x_1, x_{m-1}), D(x_0, x_1, x_{m-1})\} \leq \lambda K. \tag{3.2}
\]

Again using (3.1),

\[
D(x_2, x_3, x_m) \leq \lambda \max \{D(x_2, x_3, x_{m-1}), D(x_1, x_2, x_{m-1})\}. \tag{3.3}
\]

Using (3.2),

\[
D(x_2, x_3, x_m) \leq \lambda \max \{D(x_2, x_3, x_{m-1}), \lambda K\}. \tag{3.4}
\]

Inequality (3.4) can be regarded as a recursion formula in \(m\). Therefore

\[
D(x_2, x_3, x_m) \leq \lambda \max \{\lambda \max \{D(x_2, x_3, x_{m-1}), \lambda K\}, \lambda K\} \leq \lambda^2 K. \tag{3.5}
\]

Assume the induction hypothesis. Then, from (3.1),

\[
D(x_{n+1}, x_{n+2}, x_m) \leq \lambda \max \{D(x_{n+1}, x_{n+2}, x_{m-1}), D(x_n, x_{n+1}, x_{m-1})\}
\]

\[
\leq \lambda \max \{D(x_n, x_{n+1}, x_{m-1}), \lambda^n K\}. \tag{3.6}
\]

Inequality (3.6) can be regarded as a recursion formula in \(m\). Therefore,

\[
D(x_{n+1}, x_{n+2}, x_m) \leq \Lambda \max \{\lambda \max \{D(x_{n+1}, x_{n+2}, x_{m-1}), \lambda^n K\}, \lambda^n K\}
\]

\[
= \max \{\lambda^2 D(x_{n+1}, x_{n+2}, x_{m-1}), \lambda^{n+2} K, \lambda^{n+1} K\}
\]

\[
= \max \{\lambda^2 D(x_{n+1}, x_{n+2}, x_{m-2}), \lambda^{n+1} K\}
\]

\[
\leq \max \{\lambda^2 \cdot \lambda \max \{D(x_{n+1}, x_{n+2}, x_{m-3}), \lambda^{n+1} K\}, \lambda^{n+1} K\}
\]

\[
= \max \{\lambda^3 D(x_{n+1}, x_{n+2}, x_{m-3}), \lambda^{n+1} K\} \leq \cdots
\]

\[
\leq \max \{\lambda^n D(x_{n+1}, x_{n+2}, x_{m-n}), \lambda^{n+1} K\}
\]

\[
\leq \max \{\lambda^n \cdot \Lambda \max \{D(x_{n+1}, x_{n+2}, x_{m-n-1}), \lambda^{n+1} K\}, \lambda^{n+1} K\}
\]

\[
= \lambda^{n+1} K.
\]

and \(\{x_n\}\) is \(D\)-Cauchy by Lemma 2.1. Since \(X\) is \(x_0\)-orbitally complete, there exists a \(p \in X\) with \(\lim x_n = p\).
In (3.1) set $x = x_n$, $z = p$ to obtain
\[
D(x_{n+1}, x_{n+1}, fp) \leq \lambda \max \{D(x_n, x_n, p), D(x_n, x_{n+1}, p)\}.
\tag{3.8}
\]
Taking the limit of (3.8) as $n \to \infty$ yields $D(p, p, fp) \leq \lambda D(p, p, p) = 0$, and $p = fp$.

To prove uniqueness, suppose that $q$ is also a fixed point of $f$. Then, from (3.1),
\[
D(p, p, q) = D(fp, fp, fq) \leq \lambda \max \{D(p, p, q), D(p, fp, q)\} = \lambda D(p, p, q),
\tag{3.9}
\]
which implies that $p = q$.

**Corollary 3.2** [2, Theorem 2.1]. Let $f$ be a selfmap of a complete and bounded $D$-metric space $X$ satisfying
\[
D(fx, fy, fz) \leq \lambda D(x, y, z)
\tag{3.10}
\]
for all $x, y, z \in X$, for some $0 \leq \lambda < 1$. Then $f$ has a unique fixed point $p$, and $f$ is continuous at $p$.

**Proof.** In (3.10) set $y = fx$ to obtain (3.1). Then, from Theorem 3.1, $f$ has a unique fixed point $p$.

To prove continuity, let $(z_n) \subset X$ with $\lim z_n = p$. From (3.10),
\[
D(p, p, fz_n) = D(fp, fp, fz_n) \leq \lambda D(p, p, z_n).
\tag{3.11}
\]
Taking the limit as $n \to \infty$ gives $\limsup D(p, p, fz_n) = 0$, and $\liminf D(p, p, fz_n) = 0$ which implies that $\lim fz_n = p = fp$, and $f$ is continuous at $p$.

**Corollary 3.3** [2, Corollary 1.1]. Let $f$ be a selfmap of a complete and bounded $D$-metric space satisfying the condition that there exists a positive integer $q$ such that
\[
D(f^q x, f^q y, f^q z) \leq \lambda D(x, y, z)
\tag{3.12}
\]
for all $x, y, z \in X$, for some $0 \leq \lambda < 1$. Then $f$ has a unique fixed point $p$, and $f$ is $f$-orbitally continuous at $p$.

**Proof.** Define $T = f^q$. Then (3.12) reduces to (3.10), and $T$ has a unique fixed point $p$ by Corollary 3.2; i.e., $p =Tp = f^q p$. Thus $fp = f^{q+1} p = T(fp)$, and $fp$ is also a fixed point of $T$. Uniqueness implies that $fp = p$, and $p$ is a fixed point of $f$. Condition (3.12) implies the uniqueness of $p$ as a fixed point of $f$.

For the continuity, let $(z_n) \subset O(f)$, with $\lim z_n = p$. From (3.12),
\[
D(f^q p, f^q p, f^q z_n) \leq \lambda D(p, p, z_n).
\tag{3.13}
\]
Taking the limit as $n \to \infty$ shows that $\lim f^q z_n = p = f^q p$, and $f^q$ is $f$-orbitally continuous at $p$. But, since each $z_n \in O(f)$, $\lim f^q z_n = \lim f z_{n+q-1}$, and $f$ is $f$-orbitally continuous at $p$.

**Corollary 3.4.** Let $f$ be a selfmap of $X$, $X$ an $f$-orbitally bounded and complete $D$-metric space satisfying
\[
D(fx, fy, fz) \leq \alpha \left[ \frac{1 + D(x, fx, z)}{1 + D(x, y, z)} \right] D(y, fy, z) + \beta D(x, y, z)
\tag{3.14}
\]
for all \( x, y, z \in X, \alpha, \beta \geq 0, \alpha + \beta < 1 \). Then \( f \) has a unique fixed point \( p \) and \( f \) is continuous at \( p \).

**Proof.** In (3.14) set \( y = fx \) to obtain

\[
D(fx, f^2x, fz) \leq \alpha D(fx, f^2x, z) + \beta D(x, fx, z)
\]

\[
\leq \lambda \max \{D(fx, f^2x, z), D(x, fx, z)\},
\]

(3.15)

where \( \lambda = \alpha + \beta < 1 \), and (3.1) is satisfied. The conclusion follows from Theorem 3.1.

To prove the continuity of \( f \) at \( p \), let \( \{z_n\} \subset X \) with \( \lim z_n = p \). In (3.14) set \( x = z = p, y = z_n \), to obtain

\[
D(p, fz_n, p) = D(fp, fz_n, fp)
\]

\[
\leq \alpha \left[ \frac{1 + D(p, fp, p)}{1 + D(p, z_n, p)} \right] D(z_n, fz_n, p) + \beta D(p, z_n, p)
\]

\[
\leq \alpha D(z_n, fz_n, p) + \beta D(p, z_n, p).
\]

(3.16)

Taking the limsup of both sides of (3.16) as \( n \to \infty \) yields

\[
D(p, \limsup fz_n, p) \leq \alpha D(p, \limsup fz_n, p),
\]

(3.17)

which implies that \( \limsup fz_n = p \). Similarly, taking the liminf of both sides of (3.16) as \( n \to \infty \) yields

\[
D(p, \liminf fz_n, p) \leq \alpha D(p, \liminf fz_n, p),
\]

(3.18)

which implies that \( \liminf fz_n = p \). Therefore \( \lim fz_n = p = fp \), and \( f \) is continuous at \( p \).

**Corollary 3.5.** Let \( f \) be a selfmap of an \( f \)-orbitally bounded and complete \( D \)-metric space \( X \), \( q \) a fixed positive integer. Suppose that \( f \) satisfies

\[
D(f^q x, f^q y, f^q z) \leq \alpha \left[ \frac{1 + D(x, f^q x, z)}{1 + D(x, y, z)} \right] D(y, f^q y, z) + \beta D(x, y, z)
\]

(3.19)

for all \( x, y, z \in X \), where \( \alpha, \beta \geq 0, \alpha + \beta < 1 \). Then \( f \) has a unique fixed point \( p \) and \( f \) is \( f \)-orbitally continuous at \( p \).

**Proof.** Set \( T = f^q \). Then \( T \) satisfies (3.14). Therefore \( T \) has a unique fixed point at \( p \), and \( f \) is continuous at \( p \). A standard argument then verifies that \( f \) has \( p \) as a unique fixed point. As in the proof of Corollary 3.3, \( f \) is \( f \)-orbitally continuous at \( p \).

4. \( \alpha \)-condensing maps. For any set \( A \) in a \( D \)-metric space \( X \), the \( D \)-diameter of \( A, \delta(A) \), is defined by \( \delta(A) = \sup_{x, y, z \in A} D(x, y, z) \). The measure of noncompactness of a bounded set \( A \) in a \( D \)-metric space \( X \) is a nonnegative real number \( \alpha(A) \) defined by

\[
\alpha(A) = \inf \{ y > 0 : A = \bigcup_{i=1}^{n} A_i \text{ for which } \delta(A_i) \leq y \text{ for } i = 1, 2, \ldots, n \}.
\]

(4.1)
**Definition 4.1.** A selfmap $f$ of $X$ is called $\alpha$-condensing if, for any bounded set $A$ in $X$, $f(A)$ is bounded and $\alpha(f(A)) < \alpha(A)$ if $\alpha(A) > 0$.

Some authors refer to $\alpha$-condensing maps as densifying maps.

**Lemma 4.2.** Let $f : X \to X$, $X$ an $f$-orbitally bounded and complete $D$-metric space, be $\alpha$-condensing. Then $O(x)$ is compact for each $x \in X$.

**Proof.** Let $x \in X$ and define $A \subset X$ by $A = \{x, fx, f^2x, \ldots\} = \{x\} \cup f(A)$. Therefore, if $A$ is not precompact, then $\alpha(A) = \alpha(f(A)) < \alpha(A)$, a contradiction. Therefore $\bar{A} = \overline{O(x)}$ is compact, since $\bar{A}$ is a complete $D$-metric space.

Define $\delta(x, y, z) = \delta(O(x) \cup O(y)O(z))$

**Theorem 4.3.** Let $f$ be a continuous compact selfmap of a bounded $D$-metric space $X$, satisfying

\[D(f^r x, f^s y, f^t z) < \delta(x, y, z) \text{ for each } x, y, z \in X, \text{ with two of } \{x, y, z\} \text{ distinct},\]

(4.3)

where $r, s, \text{ and } t$ are fixed positive integers. Then $f$ has a unique fixed point in $X$.

**Proof.** Since $f$ is compact, there exists a compact subset $Y$ of $X$ containing $fX$. Then $fY \subset Y$ and $A := \cap_{n=1}^{\infty} f^n Y$ is a nonempty compact $f$-invariant subset of $X$ which is mapped by $f$ onto itself. $A$ has the same properties with respect to $f^r, f^s, \text{ and } f^t$.

Suppose that $\delta(A) > 0$. Since $A$ is compact there exist $x, y, z \in A$ such that $\delta(A) = D(x, y, z)$. Since $fA = A$, there exist $x', y', \text{ and } z'$ in $A$ such that $x = f^r x', y = f^s y', \text{ and } z = f^t z'$. Then, from (4.3),

\[\delta(A) = D(x, y, z) = D(f^r x', f^s y', f^t z') < \delta(x, y, z) = \delta(A),\]

(4.4)

a contradiction. Therefore $A$ consists of a single point, which is a fixed point of $f$.

Suppose $p$ and $q$ are fixed points of $f$, $p \neq q$. Then, from (4.3),

\[0 < D(p, p, q) = D(f^r p, f^s p, f^t q) < D(p, p, q),\]

(4.5)

a contradiction. Therefore the fixed point is unique.

**Corollary 4.4 [8, Theorem 2].** Let $X$ be a compact $D$-metric space, $f$ a continuous selfmap of $X$ satisfying

\[D(fx, fy, fz) < \max\{D(x, y, z), D(x, fx, z), D(y, fy, z), D(x, fy, z), D(y, fx, z), D(y, fz)\}D(p, p, q)\]

(4.6)

for all $x, y, z \in X$ with $x \neq fx$, $y \neq fy$, or $z \neq fz$. Then $f$ has a unique fixed point $p$ in $X$. 

PROOF. Inequality (4.6) implies that $D(fx, fy, fz) < \delta(x, y, z)$, and the existence and uniqueness of a fixed point $p$ follows from Theorem 4.3.

For continuity, let $\{z_n\} \subset X$ with $z_n \neq p$ for each $n$ and $\lim z_n = p$. From (4.6)

$$D(p, p, fz_n) = D(fp, fp, fz_n) < D(p, fp, z_n). \tag{4.7}$$

Taking the limit as $n \to \infty$ implies that $f$ is continuous at $p$. \hfill \Box

THEOREM 4.5. Let $f$ be an $f$-orbitally continuous $\alpha$-condensing selfmap of a complete bounded $D$-metric space $X$. Let $a \in X$. If (4.3) holds on $\overline{O(a)}$, then $f$ has a unique fixed point $p \in \overline{O(a)}$.

PROOF. From Lemma 4.2, $\overline{O(a)}$ is compact. Since $f$ is a continuous $\alpha$-condensing selfmap of $\overline{O(a)}$, $f$ is compact. Now apply Theorem 4.3. \hfill \Box

COROLLARY 4.6. Let $f$ be a continuous $\alpha$-condensing selfmap of a complete bounded $D$-metric space $X$ satisfying (4.6) for all $x, y, z \in X$ with $x \neq fx$, $y \neq fy$, or $z \neq fz$. Then $f$ has a unique fixed point $p$ in $X$.

As in the proof of Corollary 4.4, $D(fx, fy, fz) < \delta(x, y, z)$ and the result follows from Theorem 4.5.

THEOREM 4.7. Let $f$ be a selfmap of a $D$-metric space $X$. Suppose that there exists a point $a \in X$ with $\overline{O(a)}$ bounded and complete. Suppose that $f$ is $f$-orbitally continuous and $\alpha$-condensing on $\overline{O(a)}$ and satisfies (4.3) for each $x, y, z \in \overline{O(a)}$ with two of $\{x, y, z\}$ distinct, and $x \neq fx$, $y \neq fy$, $z \neq fz$. Then $f$ has a fixed point in $\overline{O(a)}$.

PROOF. By Lemma 4.2 $\overline{O(a)}$ is compact. If there exists some integer $n$ for which $f^na = f^{n+1}a$, then $f$ has a fixed point in $\overline{O(a)}$. Assume that $f^na \neq f^{n+1}a$ for each $n$. Note that $f$, restricted to $\overline{O(a)}$ is a continuous compact selfmap of $\overline{O(a)}$. Suppose that $u \neq fu$ for each cluster point $u$ of $\overline{O(a)}$. Then $f$ satisfies condition (4.3) for all $x, y, z \in \overline{O(a)}$, with two of $\{x, y, z\}$ distinct. Therefore, by Theorem 4.3, $f$, restricted to $\overline{O(a)}$, has a unique fixed point $p \in \overline{O(a)}$. This contradicts the assumption that $u \neq fu$ for each cluster point $u$ of $\overline{O(a)}$. Therefore $u = fu$ for some cluster point $u \in \overline{O(a)}$.

The proofs of Theorems 4.3, 4.5, and 4.7 are very similar to their metric space counterparts in [6] and [7], but have been given here for completeness.

The following results are proved using the proof technique analogous to the corresponding metric space theorems. \hfill \Box

THEOREM 4.8. Let $f$ be a selfmap of $X$, an $f$-orbitally bounded and complete $D$-metric space. Suppose that $f$ is $\alpha$-condensing, $f$-orbitally continuous and satisfies

$$D(fx, fy, fz) < \alpha \left[ \frac{1 + D(x, fx, z)}{1 + D(x, y, z)} \right] D(y, fy, z) + \beta D(x, y, z) = M(x, y, z) \tag{4.8}$$

for all $x, y, z \in X$ with $x \neq fx$, $y \neq fy$, $z \neq fz$, where $\alpha, \beta > 0$, $\alpha + \beta \leq 1$. Then $f$ has a unique fixed point $p \in X$ and $f$ is continuous at $p$.

PROOF. If $\alpha + \beta < 1$, the result follows from Corollary 3.4. Therefore we assume that $\alpha + \beta = 1$. Let $x_0 \in X$ and define $x_{n+1} = fx_n$, $n \geq 0$. From Lemma 4.2 it follows that $\overline{O(x_0)}$ is compact. Obviously $f : \overline{O(x_0)} \to \overline{O(x_0)}$. \hfill \Box
**Case I.** There exists some \( x, y, z \in \overline{O(x_0)} \) for which \( M = 0 \). Then \( y = fy = z = x \), and \( y \) is a fixed point of \( f \). Inequality (4.8) implies uniqueness.

**Case II.** \( M \neq 0 \) for all \( x, y, z \in \overline{O(x_0)} \). Define a function \( F : (\overline{O(x_0)})^3 \to (0, \infty) \) by

\[
F(x, y, z) = \frac{D(fx, fy, fz)}{M(x, y, z)}.
\]

The function \( F \) is well defined on \( (\overline{O(x_0)})^3 \) since \( M \neq 0 \) on \( \overline{O(x_0)} \).

Since \( F \) is continuous on \( \overline{O(x_0)} \), it attains its maximum value at some point \((u, v, w)\) \( \in \overline{O(x_0)} \). We call this maximum value \( c \). From (4.8) it follows that \( 0 < c < 1 \). Therefore

\[
D(fx, fy, fz) \leq cM(x, y, z)
\]

for all \( x, y, z \in \overline{O(x_0)} \), where \( \alpha' = c\alpha > 0 \), \( \beta' = c\beta > 0 \), and \( \alpha' + \beta' = c(\alpha + \beta) < 1 \). Since \( \overline{O(x_0)} \) is compact, it is bounded and complete. The result follows from Corollary 3.4.

\[\square\]

**Corollary 4.9.** Let \( f \) be a selfmap of a complete and \( f \)-orbitally bounded \( D \)-metric space. Suppose that \( f \) is \( \alpha \)-condensing and \( f \)-orbitally continuous. Let \( q \) be a positive integer. Suppose that \( f \) satisfies

\[
D(f^q x, f^q y, f^q z) < \alpha \left[ \frac{1 + D(x, f^q x, z)}{1 + D(x, y, z)} \right] D(y, f^q y, z) + \beta D(x, y, z)
\]

for all \( x, y, z \in X \) for which the right-hand side of (4.11) is not zero, where \( \alpha, \beta > 0 \), \( \alpha + \beta \leq 1 \). Then \( f \) has a unique fixed point \( p \) and \( f \) is \( f \)-orbitally continuous at \( p \).

**Proof.** Set \( T = f^q \). Then \( T \) satisfies (4.8), and the existence and uniqueness of the fixed point \( p \), for \( T \), follows from Theorem 4.8. It then follows that \( p \) is the unique fixed point for \( f \). The continuity argument is the same as that used in the proof of Corollary 3.3.

\[\square\]

**Corollary 4.10.** Let \( f \) be a continuous selfmap of a compact \( D \)-metric space satisfying (4.8). Then \( f \) has a unique fixed point \( p \), and \( f \) is continuous at \( p \).

This result is an immediate consequence of Theorem 4.8.

Corollary 4.10 includes [3, Theorem 2.2] as a special case.

**References**


DHAGE AND PATHAN: MATHEMATICAL RESEARCH CENTRE, MAHATMA RANDHI MAHAVIDYALAYA, AHMEDPUR-413 515, INDIA

RHOADES: DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405-4301, USA

E-mail address: rhoades@indiana.edu