ALMOST AUTOMORPHIC SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. We discuss the conditions under which bounded solutions of the evolution equation $x'(t) = Ax(t) + f(t)$ in a Banach space are almost automorphic whenever $f(t)$ is almost automorphic and $A$ generates a $C_0$-group of strongly continuous operators. We also give a result for asymptotically almost automorphic solutions for the more general case of $x'(t) = Ax(t) + f(t, x(t))$.

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1. Introduction. Let $A$ generate a $C_0$-group of strongly continuous operators $T(t)$, $t \in \mathbb{R}$ on a Banach space $X$. Let $f \in L^\infty(\mathbb{R}; X)$. A basic unsolved problem is: what is the structure of bounded (on $\mathbb{R}$) mild solutions of $x'(t) = Ax(t) + f(t)$? Classically results go back to Ordinary Differential Equations (when dimension of $X$ is finite), and one sought solutions $x(t)$ such that $x(t) - y(t) \to 0$ as $t \to \infty$, when either $y(t)$ is a constant or a periodic function of time. In the evolution context of $x' = Ax + f$, much has been written on asymptotically constant or periodic solutions. Several authors extended these ideas to almost periodic solutions (when $f$ is almost periodic). Our main result (Theorem 1.6) is inspired by the interesting work of Goldstein [3]. We are actually concerned with the more general case of almost automorphic, and when bounded solutions are almost automorphic. We also give a new result (Theorem 1.7) concerning mild solutions of the equation $x'(t) = Ax(t) + f(t, x(t))$ which approach almost automorphic functions at infinity under specific conditions on the function $f(t, x)$. See also [6] for another comparable situation.

Let $X$ be a Banach space equipped with the topology norm and $\mathbb{R} = (-\infty, \infty)$ the set of real numbers. Let us first recall some definitions.

**Definition 1.1** (Bochner). A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if and only if, from any sequence of real numbers $(s_n^r)_{n=1}^\infty$, we can subtract a subsequence $(s_n)_{n=1}^\infty$ such that: $\lim_{n \to \infty} f(t + s_n) = g(t)$ exists for each real number $t$, and $\lim_{n \to \infty} g(t - s_n) = f(t)$ for each $t$.

**Definition 1.2** [4]. A continuous function $f : \mathbb{R}^+ \to X$ is said to be asymptotically almost automorphic if and only if there exists an almost automorphic function $g : \mathbb{R} \to X$ and a continuous function $h : \mathbb{R}^+ \to X$ with $\lim_{t \to \infty} \|h(t)\| = 0$ and such that $f(t) = g(t) + h(t)$ for each $t \in \mathbb{R}^+$. 
**Definition 1.3.** A Banach space $X$ is said to be perfect if and only if every bounded function $u : \mathbb{R} \to X$ with an almost automorphic derivative $u'(t)$ is necessarily almost automorphic.

**Remark 1.4.** Uniformly convex Banach spaces are nice examples of perfect Banach spaces (see [10, Theorem 1.4]).

We consider the evolution equation

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}. \quad (1.1)$$

**Theorem 1.5.** Let $X$ be a perfect Banach space. Let $A$ be a bounded linear operator $X \to X$ and $f : \mathbb{R} \to X$ an almost automorphic function. Then any bounded strong solution of (1.1) is almost automorphic if we assume that there exists a finite-dimensional subspace $X_1$ of $X$ such that

- $A x(0) \in X_1,$
- $(e^{tA} - I)f(s) \in X_1$ for any $s, t \in \mathbb{R},$
- $e^{tA}u \in X_1$ for any $t \in \mathbb{R}$ and for any $u \in X_1.$

**Proof.** Let $P$ be the projection of $X$ onto $X_1;$ such $P$ always exists (cf. [7]) and possesses the following properties:

1. $X = X_1 \oplus \ker(P), \text{ where } \ker(P) \text{ is the kernel of the operator } P,$
2. $P$ is bounded on $X.$

If we put $Q = I - P,$ then it is easy to verify that $Q^2 = Q$ on $X$ and $Qu = 0$ for any $u \in X_1.$ Now if $x(t)$ is a bounded solution of (1.1), then we can write it as

$$x(t) = x_1(t) + x_2(t) \quad (1.2)$$

with $x_1(t) = Px(t) \in X_1$ and $x_2(t) = Qx(t) \in \ker(P).$

Since $x(t)$ is bounded on $\mathbb{R},$ it is clear that both $x_1(t)$ and $x_2(t)$ are also bounded on $\mathbb{R}.$ On the other hand, we have

$$x'(t) = x_1'(t) + x_2'(t) = Ax_1(t) + Ax_2(t) + Pf(t) + Qf(t), \quad t \in \mathbb{R}. \quad (1.3)$$

But $x(t)$ has the well-known Lagrange representation:

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A}f(s) \, ds$$

$$= e^{tA}x(0) + \int_0^t f(s) \, ds + \int_0^t (e^{(t-s)A} - I)f(s) \, ds. \quad (1.4)$$

By assumption $(\beta),$ we deduce that $\int_0^t (e^{(t-s)A} - I)f(s) \, ds$ is in $X_1,$ so that if we apply $Q$ to both sides of (1.4), we get

$$x_2(t) = Qe^{tA}x(0) + Q \int_0^t f(s) \, ds = Qe^{tA}x(0) + \int_0^t Qf(s) \, ds, \quad (1.5)$$

consequently

$$x_2'(t) = Qe^{tA}Ax(0) + Qf(t) = Qf(t) \quad (1.6)$$

using conditions $(\alpha)$ and $(\gamma).$
It is clear that $Qf(t)$ and thus $x'_2(t)$ is almost automorphic (see [9, page 586]). Since $x_2(t)$ is bounded, then it is almost automorphic for we are in a perfect Banach space.

Now if we apply $P$ to both sides of (1.3), we get in the finite-dimensional space $X_1$ the differential equation

$$x'_1(t) = PAx_1(t) + PAx_2(t) + P^2f(t) + PQf(t), \quad t \in \mathbb{R}. \quad (1.7)$$

Since the function $g(t) = P^2f(t) + PQf(t)$ is almost automorphic and $PA$ is a bounded linear operator, we deduce that $x_1(t)$ is almost automorphic [9, Theorem 3]. Finally, $x(t)$ is almost automorphic as the sum of two almost automorphic functions.

Theorem 1.5 can be generalized to the case of unbounded operator $A$ as follows.

**Theorem 1.6.** In a perfect Banach space $X$, let $A$ generate a $C_0$-group of strongly continuous linear operators $T(t), t \in \mathbb{R}$. Assume that there exists a finite-dimensional subspace $X_1$ of $X$ such that:

1. $Ax(0) \in X_1$,
2. $(T(t) - I)f(s) \in X_1$ for any $s, t \in \mathbb{R}$,
3. $T(t)u \in X_1$ for any $t \in \mathbb{R}$ and any $u \in X_1$.

Then every bounded solution of (1.1) is almost automorphic.

**Proof.** We just follow the proof of Theorem 1.5 with the appropriate modifications. Here solutions are written as $x(t) = T(t)x(0) + \int_0^t T(t-s)f(s)ds$.

We return now to a general (not necessarily perfect) Banach space $X$. We state and prove the following theorem.

**Theorem 1.7.** Let $A$ be a (possibly unbounded) linear operator which is the generator of a $C_0$-group of strongly continuous linear operators $T(t), t \in \mathbb{R}$ such that $T(t)x : \mathbb{R} \to X$ is almost automorphic for each $x \in X$. Consider the differential equation

$$x'(t) = Ax(t) + f(t, x(t)), \quad (1.8)$$

where $f(t, x) : \mathbb{R} \times X \to X$ is strongly continuous with respect to jointly $t$ and $x$ and such that $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$ for any $t \in \mathbb{R}$, $x, y \in X$, and $\int_0^\infty \|f(t, 0)\|dt < \infty$.

Then every mild solution $x(t)$ of (1.8) with $\int_0^\infty \|x(t)\|dt < \infty$ is asymptotically almost automorphic.

**Proof.** Let $x : \mathbb{R}^+ \to X$ be a mild solution of (1.8). Then we have

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(s, x(s))ds. \quad (1.9)$$

We claim that $\int_0^\infty T(-s)f(s, x(s))ds$ exists in $X$ (in Bochner’s sense). Indeed, since $T(t)$ is almost automorphic for each $x \in X$, then

$$\sup_{t \in \mathbb{R}} \|T(t)x\| < \infty \quad \text{for each } x \in X. \quad (1.10)$$

Consequently

$$\sup_{t \in \mathbb{R}} \|T(t)\| = M < \infty, \quad (1.11)$$
by the uniform boundedness principle. Let us write
\[ \int_0^\infty T(-s)f(s,x(s))\,ds = \int_0^\infty T(-s)(f(s,x(s))-f(s,0))\,ds + \int_0^\infty T(-s)f(s,0)\,ds, \tag{1.12} \]
then we get the inequality
\[ \left\| \int_0^\infty T(-s)f(s,x(s))\,ds \right\| \leq M \left( L \int_0^\infty \| x(s)\|\,ds + \int_0^\infty \| f(s,0)\|\,ds \right) < \infty. \tag{1.13} \]
Now the continuous function \( F: \mathbb{R} \to X \) defined by
\[ F(t) = \int_0^\infty T(t-s)f(s,x(s))\,ds = T(t) \int_0^\infty T(-s)f(s,x(s))\,ds \tag{1.14} \]
is almost automorphic; therefore \( V(t) = T(t)x(0) + F(t) \) is also almost automorphic. Let us consider the continuous function \( W: \mathbb{R}^+ \to X \)
\[ W(t) = -\int_t^\infty T(t-s)f(s,x(s))\,ds. \tag{1.15} \]
If we use the same computation as for \( F(t) \) in (1.14), we get
\[ \| W(t)\| \leq M \left( L \int_t^\infty \| x(s)\|\,ds + \int_t^\infty \| f(s,0)\|\,ds \right) \tag{1.16} \]
which shows that \( \lim_{t \to \infty} \| W(t)\| = 0. \)
Finally \( x(t) = V(t) + W(t), \ t \in \mathbb{R}^+ \) is asymptotically almost automorphic.

\[ \text{Remark 1.8.} \] (1) An example of Theorem 1.5 (occurring in Sturm-Liouville theory, for instance) is when \( X \) is a Hilbert space and \( A \varphi_n = \lambda_n \varphi_n \) for \( \{ \varphi_n : n = 1,2,\ldots \} \) an orthonormal basis and \( |\Re(\lambda_n)| \leq M \) for all \( n \). For \( X_1 \), one may take \( X_1 = \text{span}\{ \varphi_1,\ldots, \varphi_N \} \) (for any \( N \)) and assume \( f \in L^\infty(\mathbb{R},X_1) \).
(2) An example of operator \( A \) satisfying the hypothesis of Theorem 1.7 is the above example with \( A^* = -A, \ i.e., \ M = 0. \)

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