A NOTE ON \((gDF)\)-SPACES

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ABSTRACT. Certain locally convex spaces of scalar-valued mappings are shown to be finite-dimensional.

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1. Introduction. Radenovič [6], generalizing a result of Iyahen [2], has shown that if \(E\) is a Banach space and \((E, \sigma(E, E'))\) (or \((E', \sigma(E', E))\)) is a \((DF)\)-space [1], then \(E\) is finite-dimensional. His result has been extended to arbitrary locally convex spaces by Krassowska and Śliwa [3].

In [4, 5], \((DF)\)-spaces have been generalized as follows: a locally convex space \((E, \tau)\) is a \((gDF)\)-space if

(a) \((E, \tau)\) has a fundamental sequence \((B_n)_{n \in \mathbb{N}}\) of bounded sets, and

(b) \(\tau\) is the finest locally convex topology on \(E\) that agrees with \(\tau\) on each \(B_n\).

In this note, we prove that if an arbitrary vector space of scalar-valued mappings is a \((gDF)\)-space under the locally convex topology of pointwise convergence, then it is finite-dimensional. As a consequence, the above-mentioned theorem of Krassowska and Śliwa readily follows.

2. The result. Throughout this note, all vector spaces under consideration are vector spaces over a field \(\mathbb{K}\) which is either \(\mathbb{R}\) or \(\mathbb{C}\). In our result, \(E\) denotes an arbitrary set and \(H\) denotes a subspace of the vector space of all mappings from \(E\) into \(\mathbb{K}\). We consider on \(H\) the separated locally convex topology of pointwise convergence and represent by \(H'\) the topological dual of \(H\).

**Theorem 2.1.** The following conditions are equivalent:

(a) \(H\) is a finite-dimensional vector space;

(b) \(H\) is a \((DF)\)-space;

(c) \(H\) is a \((gDF)\)-space.

**Proof.** It is clear that (a) implies (b) and (b) implies (c) (every \((DF)\)-space is a \((gDF)\)-space).

Suppose that condition (c) holds. If \(H\) is infinite-dimensional, there exists a countable linearly independent subset \(\{\varphi_n; n \in \mathbb{N}\}\) of \(H'\). Let \((B_n)_{n \in \mathbb{N}}\) be an increasing fundamental sequence of bounded subsets of \(H\). Then, \((B_n^0)_{n \in \mathbb{N}}\) is a decreasing sequence of neighborhoods of zero in \((H', \beta(H', H))\) forming a fundamental system...
of neighborhoods of zero in \((H',\beta(H',H))\). For each \(n \in \mathbb{N}\), fix an \(\alpha_n > 0\) such that \(\alpha_n \varphi_n \in B_0^H\); then \((\alpha_n \varphi_n)_{n \in \mathbb{N}}\) converges to zero in \((H',\beta(H',H))\). By [5, Theorem 1.1.7], the set \(\Gamma = \{\alpha_n \varphi_n; n \in \mathbb{N}\}\) is equicontinuous. Hence, there exist \(x_1, \ldots, x_m \in E\) and there exists an \(\alpha > 0\) such that the relations

\[
f \in H, \quad |f(x_1)| \leq \alpha, \ldots, |f(x_m)| \leq \alpha, \quad \varphi \in \Gamma
\]  

(2.1)

imply

\[
|\varphi(f)| \leq 1.
\]  

(2.2)

For each \(i = 1, \ldots, m\), let \(\delta_i \in H'\) be given by \(\delta_i(f) = f(x_i)\) for \(f \in H\), and put \(F = \{\delta_1, \ldots, \delta_m\}\). We claim that \(\Gamma \subset [F]\), where \([F]\) is the finite-dimensional vector space generated by \(F\). Indeed, let \(\varphi \in \Gamma\) and take an \(f \in H\) such that \(\delta_1(f) = \cdots = \delta_m(f) = 0\). Then, for all \(\lambda \in \mathbb{K}\),

\[
|(\lambda f)(x_1)| = |\delta_1(\lambda f)| = 0 \leq \alpha, \ldots, |(\lambda f)(x_m)| = |\delta_m(\lambda f)| = 0 \leq \alpha.
\]  

(2.3)

Consequently, \(|\varphi(\lambda f)| = |\lambda||\varphi(f)| \leq 1\). By the arbitrariness of \(\lambda, \varphi(f) = 0\). By [7, Lemma 5, Chapter II], \(\varphi \in [F]\). Therefore the vector space generated by the set \(\{\varphi_n; n \in \mathbb{N}\}\) is finite-dimensional, which contradicts the choice of \((\varphi_n)_{n \in \mathbb{N}}\). This completes the proof of the theorem. \(\square\)

**Remark 2.2.** The theorem of Krassowska and Śliwa mentioned at the beginning of this note follows from Theorem 2.1. In fact, let \(E\) be a separated locally convex space. If \((E', \sigma(E',E))\) is a \((DF)\)-space, then \(E'\) is finite-dimensional by Theorem 2.1, and so \(E\) is finite-dimensional. Hence, \(E\) is finite-dimensional if \((E, \sigma(E,E'))\) is a \((DF)\)-space.

**References**


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