A NOTE ON A RESULT OF SINGH AND KULKARNI

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Abstract. We prove that if \( f \) is a transcendental meromorphic function of finite order and \( \sum \delta(a,f) + \delta(\infty,f) = 2 \), then

\[
K(f^{(k)}) = \frac{2k(1-\delta(\infty,f))}{1+k-k\delta(\infty,f)},
\]

where

\[
K(f^{(k)}) = \lim_{r \to \infty} \frac{N(r,1/f^{(k)}) + N(r,f^{(k)})}{T(r,f^{(k)})}.
\]

This result improves a result by Singh and Kulkarni.

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1. Introduction and the main result. Let \( f(z) \) be a meromorphic function in the complex plane. We use the following notations of value distribution theory (see [2])

\[
T(r,f), m(r,f), N(r,f), N(r,f),...
\]

and denote by \( S(r,f) \) a function with the property that \( S(r,f) = o(T(r,f)) \), \( r \to \infty \) (outside an exceptional set of finite linear measure, if \( f \) is of infinite order). The Nevanlinna’s deficiency of \( f \) with respect to a finite complex number \( a \) is defined by

\[
\delta(a,f) = \lim_{r \to \infty} \frac{m(r,1/(f-a))}{T(r,f)}.
\]

If \( a = \infty \), then one should replace \( m(r,1/(f-a)) \) in the above formula by \( m(r,f) \).

The well known Nevanlinna’s deficiency relation states that

\[
\sum_{a \neq \infty} \delta(a,f) + \delta(\infty,f) \leq 2.
\]

If the above inequality holds, then we say that \( f \) has maximum deficiency sum.

In [3], Singh and Kulkarni proved the following result.

**Theorem 1.1.** Suppose that \( f \) is a transcendental meromorphic function of finite order and \( \sum \delta(a,f) + \delta(\infty,f) = 2 \), then

\[
\frac{1-\delta(\infty,f)}{2-\delta(\infty,f)} \leq K(f') \leq \frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)},
\]
where
\[ K(f) = \lim_{r \to \infty} \frac{N(r, (1/f')) + N(r, f')}{T(r, f')}. \] (1.5)

In this note, we prove the following.

**Theorem 1.2.** Suppose that \( f \) is a transcendental meromorphic function of finite order and \( \sum a \neq \delta(a, f) + \delta(\infty, f) = 2 \), then
\[ K(f^{(k)}) = \frac{2k(1 - \delta(\infty, f))}{1 + k - k\delta(\infty, f)}, \] (1.6)

where
\[ K(f^{(k)}) = \lim_{r \to \infty} \frac{N(r, 1/f^{(k)}) + N(r, f^{(k)})}{T(r, f^{(k)})}. \] (1.7)

2. An important lemma

**Lemma 2.1** [1]. Let \( f(z) \) be a transcendental meromorphic function, then for each positive number \( \epsilon \) and each positive integer \( k \), we have
\[ kN(r, f) \leq N(r, 1/f^{(k)}) + N(r, f) + \epsilon T(r, f) + S(r, f). \] (2.1)

**Proof of Theorem 1.2.** First, we prove that
\[ \lim_{r \to \infty} \frac{T(r, f^{(k)})}{T(r, f)} = 1 + k - k\delta(\infty, f), \quad r \to \infty. \] (2.2)

Without loss of generality, we assume that \( f \) has infinitely many finite deficient values \( a_1, a_2, \ldots \). It follows from Littlewood’s inequality
\[ \sum_{n=1}^{p} m \left( r, \frac{1}{f - a_n} \right) \leq m \left( r, \frac{1}{f} \right) + S(r, f) \leq T(r, f) + \overline{N}(r, f) + S(r, f), \] (2.3)

that
\[ \sum_{n=1}^{p} \delta(a_n, f) \leq 1 + \lim_{r \to \infty} \overline{N}(r, f) \leq 1 + \lim_{r \to \infty} \overline{N}(r, f) = 2 - \delta(\infty, f). \] (2.4)

By the assumption, we have
\[ \sum_{n=1}^{\infty} \delta(a_n, f) = 2 - \delta(\infty, f). \] (2.5)

Let \( p \to \infty \) in (2.4) and use (2.5) to obtain
\[ \lim_{r \to \infty} \frac{\overline{N}(r, f)}{T(r, f)} = \lim_{r \to \infty} \frac{N(r, f)}{T(r, f)} = 1 - \delta(\infty, f). \] (2.6)
Replacing $f'$ in (2.3) by $f^{(k)}$, we get

$$\sum_{n=1}^{p} m\left( r, \frac{1}{f-a_n} \right) \leq m\left( r, \frac{1}{f^{(k)}} \right) + S(r, f) \leq T\left( r, f^{(k)} \right) - N\left( r, \frac{1}{f^{(k)}} \right) + S(r, f).$$

(2.7)

It follows from (2.7) and (2.1) that

$$\sum_{n=1}^{p} m\left( r, \frac{1}{f-a_n} \right) \leq T\left( r, f^{(k)} \right) + N(r, f) - kN(r, f) + \epsilon T(r, f) + S(r, f).$$

(2.8)

Consequently, because of (2.6), we have

$$\lim_{r \to \infty} \frac{T\left( r, f^{(k)} \right)}{T(r, f)} \geq (k-1)(1-\delta(\infty, f)) + \sum_{n=1}^{p} \delta(a_n, f) - \epsilon.$$  (2.9)

Now, let $p \to \infty$ and $\epsilon \to 0$ and use (2.5) to obtain

$$\lim_{r \to \infty} \frac{T\left( r, f^{(k)} \right)}{T(r, f)} \geq 1 + k - k\delta(\infty, f).$$  (2.10)

On the other side,

$$T\left( r, f^{(k)} \right) \leq T(r, f) + kN(r, f) + S(r, f).$$  (2.11)

Therefore, because of (2.6),

$$\lim_{r \to \infty} \frac{T\left( r, f^{(k)} \right)}{T(r, f)} \leq 1 + k - k\delta(\infty, f).$$  (2.12)

Equation (2.2) follows from the above estimates.

Next, we prove that

$$\lim_{r \to \infty} \frac{N(r, 1/f^{(k)})}{T\left( r, f^{(k)} \right)} = \frac{(k-1)(1-\delta(\infty, f))}{1 + k - k\delta(\infty, f)}.$$  (2.13)

From the first inequality of (2.7), we have

$$\lim_{r \to \infty} \frac{m\left( r, 1/f^{(k)} \right)}{T(r, f)} \geq \sum_{n=1}^{p} \delta(a_n, f).$$  (2.14)

Consequently, if we let $p \to +\infty$ and use (2.5), we get

$$\lim_{r \to \infty} \frac{m\left( r, 1/f^{(k)} \right)}{T(r, f)} \geq 2 - \delta(\infty, f).$$  (2.15)

On the other side, from (2.1) and (2.7), we have

$$m\left( r, \frac{1}{f^{(k)}} \right) \leq T\left( r, f^{(k)} \right) - N\left( r, \frac{1}{f^{(k)}} \right) + S(r, f) \leq T(r, f) + kN(r, f) - N\left( r, \frac{1}{f^{(k)}} \right) + S(r, f) \leq T(r, f) + N(r, f) + \epsilon T(r, f) + S(r, f),$$  (2.16)
hence,
\[
\lim_{r \to \infty} \frac{m(r, 1/f(k))}{T(r, f)} \leq 2 - \delta(\infty, f) + \epsilon, \tag{2.17}
\]
if we let \( \epsilon \to 0 \), we get
\[
\lim_{r \to \infty} \frac{m(r, 1/f(k))}{T(r, f)} \leq 2 - \delta(\infty, f). \tag{2.18}
\]
Thus, from (2.15) and (2.18), we obtain
\[
\lim_{r \to \infty} \frac{m(r, 1/f(k))}{T(r, f)} = 2 - \delta(\infty, f). \tag{2.19}
\]
Hence, from (2.2), (2.18), and (2.19), we have
\[
\lim_{r \to \infty} \frac{N(r, f(k))}{T(r, f(k))} = 1 - \lim_{r \to \infty} \frac{m(r, 1/f(k))}{T(r, f(k))} \lim_{r \to \infty} \frac{T(r, f)}{T(r, f(k))} \tag{2.20}
= 1 - \frac{2 - \delta(\infty, f)}{1 + k - k\delta(\infty, f)} = \frac{(k - 1)(1 - \delta(\infty, f))}{1 + k - k\delta(\infty, f)}.
\]
Finally, from (2.2) and (2.6), we have
\[
\lim_{r \to \infty} \frac{N(r, f(k))}{T(r, f(k))} = \lim_{r \to \infty} \frac{N(r, f(k))}{T(r, f(k))} = \frac{N(r, f) + kN(r, f(k))}{T(r, f(k))} \tag{2.21}
= \frac{(k + 1)(1 - \delta(\infty, f))}{1 + k - k\delta(\infty, f)}.
\]
Therefore, we deduce, from (2.20) and (2.21), that
\[
\lim_{r \to \infty} \frac{N(r, 1/f(k)) + N(r, f(k))}{T(r, f(k))} = \frac{2k(1 - \delta(\infty, f))}{1 + k - k\delta(\infty, f)}. \tag{2.22}
\]
Thus, the proof of Theorem 1.2 is complete. \( \square \)

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**References**

