NONINCLUSION THEOREMS:
SOME REMARKS ON A PAPER BY J. A. FRIDY

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(Received 19 August 1998)

Abstract. In 1997, J. A. Fridy gave conditions for noninclusion of ordinary and of absolute summability domains. In the present note, these conditions are interpreted in a natural topological context thus giving new proofs and also explaining why one of these conditions is too weak. Also an open question posed in Fridy’s paper is answered.

Keywords and phrases. Summability matrix, summability domains inclusion, absolute summability domains inclusion.

2000 Mathematics Subject Classification. Primary 40D25, 40D09, 40C05.

1. Noninclusion for ordinary summability. Recently, J. A. Fridy [2] stated a non-inclusion theorem that can be formulated in the following way.

Theorem 1.1. Let A and B be regular matrices such that $c_A$, the summability domain of A, is included in $c_B$, the summability domain of B. Then

$$\lim_{n,k} a_{nk} = 0 \implies \lim_{n,k} b_{nk} = 0.$$  \hfill (1.1)

Here $\lim_{n,k} a_{nk} = 0$ (and, similarly, $\lim_{n,k} b_{nk} = 0$) is taken in the Pringsheim sense, that is,

$$\forall \epsilon > 0 \exists N > 0 : (n > N \text{ and } k > N) \implies |a_{nk}| < \epsilon.$$  \hfill (1.2)

Of course, this is a noninclusion theorem, since if $A$ has that limit property and $B$ does not, then $c_A \nsubseteq c_B$. The reason for the above formulation is that it emphasizes an invariance property which is stated in an invariant form in the Lemma 1.2. Therein, $e^k$ denotes the basic sequence $e^k = (0,\ldots,0,1,0,\ldots)$ with “1” in the $k$th position, and the summability domain

$$c_A = \left\{ x = (x_k) | Ax = \left( \sum_{k=1}^{\infty} a_{nk} x_k \right)_{n=1,2,\ldots} \text{ exists and converges} \right\}.$$  \hfill (1.3)

is endowed with its FK-topology (see, e.g., [3, Ch. 22]) which is given by the seminorms

$$p_r(x) := |x_r| \quad (r = 1,2,\ldots),$$

$$q_r(x) := \sup_m \left| \sum_{k=1}^{m} a_{rk} x_k \right| \quad (r = 1,2,\ldots),$$

$$p_0(x) := \|Ax\|_\infty = \sup_n \left| \sum_{k=1}^{\infty} a_{nk} x_k \right|.$$  \hfill (1.4)
Observe that all column limits of $A$ exist if and only if $\varphi := \text{span} \{ e^1, e^2, \ldots \} \subset c_A$.

**Lemma 1.2.** Let $A$ be a matrix with existing column limits. Then

\[
\left( \lim_{k \to \infty} a_{nk} = 0 \text{ for } n = 1, 2, \ldots \text{ and } \lim_{n,k} a_{nk} = 0 \right) \iff \lim_{k \to \infty} e^k = 0 \text{ in } c_A. \tag{1.5}
\]

**Proof.** Certainly, $p_r(e^k) \to 0$ as $k \to \infty$ for each $r$. Also, the condition $\lim_{k \to \infty} a_{nk} = 0$ for $n = 1, 2, \ldots$ (all row limits of $A$ are zero) is equivalent to $\lim_{k \to \infty} q_r(e^k) = 0$ for $r = 1, 2, \ldots$. Now, let $\lim_{n,k} a_{nk} = 0$ in the Pringsheim sense. Then, given $\epsilon > 0$, there exists $N_1 > 0$ such that $|a_{nk}| < \epsilon$ for $n > N_1$ and $k > N_1$. If, in addition, $\lim_{k \to \infty} a_{rk} = 0$ for $r = 1, \ldots, N_1$, then there exists $N > N_1$ such that $|a_{nk}| < \epsilon$ for $1 \leq r \leq N_1$ and all $k > N$. Thus $p_0(e^k) = \sup_n |a_{nk}| \leq \epsilon$ for all $k > N$. Hence $p_0(e^k) \to 0$ as $k \to \infty$, and $e^k \to 0$ in $c_A$ follows.

Conversely, suppose $e^k \to 0$ in $c_A$. Then, in particular, $\lim_{k \to \infty} q_r(e^k) = 0$ for $r = 1, 2, \ldots$ and $p_0(e^k) = \sup_n |a_{nk}| \to 0$ as $k \to \infty$; the former implies $\lim_{k \to \infty} a_{rk} = 0$, the latter $\lim_{n,k} a_{nk} = 0$. \hfill $\Box$

As a corollary we obtain Fridy’s result.

**Corollary 1.3.** Let $A$ be a matrix with existing column limits and with row limits zero. If $c_A \subset c_B$, then

\[
\lim_{n,k} a_{nk} = 0 \implies \lim_{n,k} b_{nk} = 0, \tag{1.6}
\]

and then, in fact, $B$ is a matrix with existing column limits and with row limits zero.

**Proof.** By the Lemma 1.2 we have $e^k \to 0$ in $c_A$. By $c_A \subset c_B$, the relative topology of $c_B$ on $c_A$ is weaker than the FK-topology of $c_A$ (see [3, Ch. 17]; hence $e^k \to 0$ in $c_B$, and, by Lemma 1.2, this means $\lim_{n,k} b_{nk} = 0$, and the row limits of $B$ are zero. \hfill $\Box$

**Remark 1.4.** In [2] it is already noticed that in Theorem 1.1 the supposition that $A$ and $B$ should be regular can be relaxed to the condition that both matrices have column and row limits zero. Corollary 1.3 is slightly more general; the existence of the column limits of $A$ is needed in order that $e^k \in c_A$ for all $k$, and hence, by $c_A \subset c_B$, the column limits of $B$ exist. It should also be remarked here that a $K$-space $E$ containing $\varphi$ is called a wedge space if $e^k \to 0$ in $E$, see G. Bennett [1, Thm. 27], asserting that $c_A$ with $\varphi \subset c_A$ is a wedge space if and only if $\lim_{k \to \infty} \sup_n |a_{nk}| = 0$.

2. Noninclusion for absolute summability. In [2] noninclusion is also considered for absolute summability; here

\[
\ell_A = \left\{ x = (x_k) \mid Ax = \left( \sum_{k=1}^{\infty} a_{nk} x_k \right) \text{ exists and } Ax \in \ell \right\}, \tag{2.1}
\]

the absolute summability domain of $A$, is concerned, where

\[
\ell = \left\{ x = (x_k) \mid \|x\|_1 := \sum_{k=1}^{\infty} |x_k| < \infty \right\}. \tag{2.2}
\]

We state the result in the following form.
THEOREM 2.1. Let $A$ be a matrix with its column sequences in $\ell$ (so that $e^k \in \ell_A$ for all $k$), and let $B$ be a matrix with $\ell_A \subset \ell_B$. If there is an index sequence $(k(j))_{j=1,2,...}$ such that

$$\lim_{j \to \infty} \sum_{n=1}^{\infty} |a_{n,k(j)}| = 0,$$

then

$$\lim_{j \to \infty} \sum_{n=1}^{\infty} |b_{n,k(j)}| = 0.$$

In [2], there is an extra condition $\ell \subset \ell_A$, but condition (2.3) is relaxed to

$$\lim_{j \to \infty} \sum_{n=\mu}^{\infty} |a_{n,k(j)}| = 0 \quad \text{for some integer } \mu,$$

and (2.4) is correspondingly weakened to

$$\lim_{j \to \infty} \sum_{n=\mu}^{\infty} |b_{n,k(j)}| = 0$$

with the same $\mu$ as in (2.5). Unfortunately, this relaxed version fails for $\mu > 1$, even if $\ell \subset \ell_A$ and the $\mu$ in (2.6) is allowed to differ from that one in (2.5). This can be seen from the following example.

EXAMPLE 2.2. For all $k = 1,2,\ldots$, define

$$a_{nk} := \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \quad \text{and} \quad b_{nk} := \frac{1}{n^2} \quad \text{for } n = 1,2,\ldots,$$

so that

$$\begin{align*}
(Ax)_n &= \begin{cases} \sum_{k=1}^{\infty} x_k, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \\
(Bx)_n &= \frac{1}{n^2} \sum_{k=1}^{\infty} x_k.
\end{align*}$$

Then, clearly,

$$\ell \subset \ell_A = \ell_B = \left\{ (x_k) \mid \sum_{k=1}^{\infty} x_k \text{ converges} \right\},$$

and

$$\lim_{j \to \infty} \sum_{n=2}^{\infty} |a_{n,k(j)}| = 0, \quad \lim_{j \to \infty} \sum_{n=\mu}^{\infty} |b_{n,k(j)}| = \sum_{n=\mu}^{\infty} \frac{1}{n^2} > 0$$

for each integer $\mu$ and each index sequence $(k(j))$.

To prove Theorem 2.1 in a topological way—similar to the proof of Corollary 1.3 (and Theorem 1.1)—we need the following lemma.

LEMMA 2.3. Let $A$ be a matrix with its column sequences in $\ell$, and let $(k(j))_{j=1,2,...}$ be an index sequence. Then

$$\lim_{j \to \infty} \sum_{n=1}^{\infty} |a_{n,k(j)}| = 0 \iff e^{k(j)} \to 0 \quad \text{in } \ell_A \text{ as } j \to \infty.$$

PROOF. The FK-topology of the FK-space $\ell_A$ is given by the seminorms $p_r,q_r$ (see above) and

$$\lim_{j \to \infty} \sum_{n=1}^{\infty} |a_{n,k(j)}| = 0 \iff e^{k(j)} \to 0 \quad \text{in } \ell_A \text{ as } j \to \infty.$$
Thus $e^{k(j)} \to 0$ in $\ell_A$ is equivalent to $p_r(e^{k(j)}) \to 0, q_r(e^{k(j)}) \to 0$ for each fixed $r = 1, 2, \ldots$ and $\|Ae^{k(j)}\|_1 = \sum_{n=1}^{\infty} |a_{n,k(j)}| \to 0$. These conditions are equivalent to the single condition $\|Ae^{k(j)}\|_1 \to 0$, since $q_r(e^{k(j)}) \leq \|Ae^{k(j)}\|_1$ and $p_r H(e^{k(j)}) = 0$ for $k(j) > r$. The lemma follows.

Theorem 2.1 is now a simple corollary of Lemma 2.3. By $\ell_A \subset \ell_B$ the FK-topology of $\ell_A$ is stronger than the relative FK-topology of $\ell_B$ on $\ell_A$. Hence $e^{k(j)} \to 0$ in $\ell_A$ implies $e^{k(j)} \to 0$ in $\ell_B$. Lemma 2.3 now yields the assertion of Theorem 2.1.

In [2] it is asked whether in Theorem 2.1 conditions (2.3) and (2.4) can be replaced by

$$\lim_{j \to \infty} \left| \sum_{n=1}^{\infty} a_{n,k(j)} \right| = 0 \quad \text{and} \quad \lim_{j \to \infty} \left| \sum_{n=1}^{\infty} b_{n,k(j)} \right| = 0,$$

respectively. The answer is negative as can be seen by the following example.

**Example 2.4.** Define $A = (a_{nk})$ and $B = (b_{nk})$ by

$$a_{nk} := \begin{cases} 
1, & \text{if } n = 1, \\
-1, & \text{if } n = 2, \text{ for } k = 1, 2, \ldots \\
0, & \text{if } n > 2, 
\end{cases}$$

and

$$b_{nk} := \begin{cases} 
1, & \text{if } n = 1, \\
0, & \text{if } n > 1, 
\end{cases}$$

so that $Ax = (\sum_{k=1}^{\infty} x_k, -\sum_{k=1}^{\infty} x_k, 0, 0, \ldots)$ and $Bx = (\sum_{k=1}^{\infty} x_k, 0, 0, \ldots)$. Then, clearly, $(\ell \subset) \ell_A = \ell_B = \{x = (x_k) | \sum_{k=1}^{\infty} x_k \text{ converges}\}$ and

$$\lim_{j \to \infty} \left| \sum_{n=1}^{\infty} a_{n,k(j)} \right| = 0, \quad \lim_{j \to \infty} \left| \sum_{n=1}^{\infty} b_{n,k(j)} \right| = 1,$$

for any index sequence $(k(j))_{j=1,2,\ldots}$.

**References**


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