ON A CLASS OF UNIVALENT FUNCTIONS

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ABSTRACT. We consider the class of univalent functions \( f(z) = z + a_3 z^3 + a_4 z^4 + \cdots \) analytic in the unit disc and satisfying \(|(z^2 f'(z)/f^2(z)) - 1| < 1\), and show that such functions are starlike if they satisfy \(|(z^2 f'(z)/f^2(z)) - 1| < (1/\sqrt{2})\).

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Let \( A \) denote the class of functions which are analytic in the unit disc \( U = \{ z : |z| < 1 \} \) and have Taylor series expansion

\[
 f(z) = z + a_2 z^2 + a_3 z^3 + \cdots ,
\]

and let \( T \) be the univalent [3] subclass of \( A \) which satisfy

\[
 |z^2 f'(z)/f^2(z) - 1| < 1, \quad z \in U.
\]

By \( T_2 \) we denote the subclass of \( T \) for which \( f''(0) = 0 \). In this paper, we prove the following theorem.

**Theorem 1.** If \( f \in T_2 \), then

(i) \( \text{Re}(f(z)/z) > 1/2, \quad z \in U \),

(ii) \( f \) is starlike in \(|z| < 1/\sqrt{2} = 0.840896\ldots\),

(iii) \( \text{Re} f'(z) > 0 \) for \(|z| < 1/\sqrt{2}\).

Items (i) and (iii) are improvements of results in [2], and (ii) is the same as in [2] but has a different proof. Furthermore, (i) and (iii) are sharp as shown by the function

\[
 f(z) = \frac{z}{1 - z^2},
\]

but the sharpness of (ii) is difficult to establish by a direct example. We also prove the following theorem which partially answers a question raised in [1].

**Theorem 2.** If \( T_{2,\mu} \) is the subclass of \( T_2 \) which satisfies

\[
 |z^2 f'(z)/f^2(z) - 1| < \mu < 1,
\]

then \( T_{2,\mu} \) is a subclass of starlike functions if \( 0 \leq \mu \leq 1/\sqrt{2} \).
We define by $B$ the class of functions $\omega$ analytic in $U$ and satisfying
\[
|\omega(z)| < 1, \quad z \in U, \quad \omega(0) = \omega'(0) = 0.
\] (5)

From Schwarz’s lemma it then follows that
\[
|\omega(z)| \leq |z|^2.
\] (6)

**Proof of Theorem 1.** If $f \in T_2$ and satisfies (2), then
\[
z^2 \frac{f'(z)}{f(z)} - 1 = \omega(z), \quad z \in U, \quad \omega \in B,
\] (7)

and by direct integration
\[
\frac{z}{f(z)} = 1 - \int_0^1 \frac{\omega(tz)}{t^2} \, dt, \quad z \in U, \quad \omega \in B.
\] (8)

From (8), we obtain
\[
\left| \frac{z}{f(z)} - 1 \right| \leq |z|^2 < 1,
\] (9)

and this gives
\[
\left| 1 - \frac{f(z)}{z} \right| \leq \left| \frac{f(z)}{z} \right|,
\] (10)

which is equivalent to $\text{Re} \, f(z)/z > 1/2$, This proves (i).

Furthermore, from (9), we obtain
\[
\left| \arg \frac{f(z)}{z} \right| \leq \sin^{-1} |z|^2.
\] (11)

From (7), we obtain
\[
z \frac{f'(z)}{f(z)} = f(z) \left(1 + \omega(z)\right)
\] (12)

and, therefore,
\[
\left| \arg \frac{zf'(z)}{f(z)} \right| = \left| \arg \frac{f(z)}{z} + \arg \left(1 + \omega(z)\right) \right| \leq 2 \sin^{-1} |z|^2.
\] (13)

This gives (ii).

In order to prove (iii), we notice that (7) yields
\[
f'(z) = \left(\frac{f(z)}{z}\right)^2 \left(1 + \omega(z)\right)
\] (14)

and, therefore,
\[
\left| \arg f'(z) \right| = \left| 2 \arg \frac{f(z)}{z} + \arg \left(1 + \omega(z)\right) \right| \leq 3 \sin^{-1} |z|^2.
\] (15)

But this is equivalent to (iii).

**Proof of Theorem 2.** If $f \in T_{2,\mu}$, we obtain from (4)
\[
z \frac{f'(z)}{f^2(z)} - 1 = \mu \omega(z), \quad \omega \in B, \quad z \in U \quad \text{and} \quad \frac{z}{f(z)} = 1 - \mu \int_0^1 \frac{\omega(tz)}{t^2} \, dt.
\] (16)
Hence

\[ z \frac{f'(z)}{f(z)} = \frac{1 + \mu \omega(z)}{1 - \mu \int_0^1 (\omega(tz)/t^2) dt}. \tag{17} \]

Now \( \text{Re} \left( \frac{f'(z)}{f(z)} \right) > 0 \) is equivalent to the condition

\[ z \frac{f'(z)}{f(z)} = \frac{1 + \mu \omega(z)}{1 - \mu \int_0^1 (\omega(tz)/t^2) dt} \neq -iT, \quad T \in \mathbb{R}. \tag{18} \]

Relation (18) is equivalent to

\[ \frac{\mu}{2} \left[ \left( \omega(z) + \int_0^1 \frac{\omega(tz)}{t^2} dt \right) + \frac{1-iT}{1+iT} \left( \omega(z) - \int_0^1 \frac{\omega(tz)}{t^2} dt \right) \right] \neq -1. \tag{19} \]

Let

\[ M = \sup_{z \in U, \omega \in B, T \in \mathbb{R}} \left| \left( \omega(z) + \int_0^1 \frac{\omega(tz)}{t^2} dt \right) + \frac{1-iT}{1+iT} \left( \omega(z) - \int_0^1 \frac{\omega(tz)}{t^2} dt \right) \right|, \tag{20} \]

then, in view of the rotation invariance of \( B \), it follows that

\[ \text{Re} \left( \frac{f'(z)}{f(z)} \right) > 0, \quad \text{if} \ \mu \leq \frac{2}{M}. \tag{21} \]

However, from (20), we notice that

\[ M \leq \sup_{z \in U, \omega \in B} \left[ \left| \omega(z) + \int_0^1 \frac{\omega(tz)}{t^2} dt \right| + \left| \omega(z) - \int_0^1 \frac{\omega(tz)}{t^2} dt \right| \right] \]

\[ \leq 2 \sup_{z \in U, \omega \in B} \left[ \sqrt{ \left| \omega(z) \right|^2 + \int_0^1 \frac{\omega(tz)}{t^2} dt \right]^2 \right] \leq 2 \sqrt{2}. \tag{22} \]

Inequality (22) follows from the parallelogram law and the last step from (6). And (21) shows that \( \mu \leq 1/\sqrt{2}. \)

\[ \square \]

**References**

