ON CHARACTERIZATIONS OF A CENTER GALOIS EXTENSION

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ABSTRACT. Let $B$ be a ring with 1, $C$ the center of $B$, $G$ a finite automorphism group of $B$, and $B^G$ the set of elements in $B$ fixed under each element in $G$. Then, it is shown that $B$ is a center Galois extension of $B^G$ (that is, $C$ is a Galois algebra over $C^G$ with Galois group $G|C \cong G$) if and only if the ideal of $B$ generated by \{c - g(c) \mid c \in C\} is $B$ for each $g \neq 1$ in $G$. This generalizes the well known characterization of a commutative Galois extension $C$ that $C$ is a Galois extension of $C^G$ with Galois group $G$ if and only if the ideal generated by \{c - g(c) \mid c \in C\} is $C$ for each $g \neq 1$ in $G$. Some more characterizations of a center Galois extension $B$ are also given.

Keywords and phrases. Galois extensions, center Galois extensions, central extensions, Galois central extensions, Azumaya algebras, separable extensions, $H$-separable extensions.

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1. Introduction. Let $C$ be a commutative ring with 1, $G$ a finite automorphism group of $C$ and $C^G$ the set of elements in $C$ fixed under each element in $G$. It is well known that a commutative Galois extension $C$ is characterized in terms of the ideals generated by \{c - g(c) \mid c \in C\} for $g \neq 1$ in $G$, that is $C$ is a Galois extension with Galois group $G$ if and only if the ideal generated by \{c - g(c) \mid c \in C\} is $C$ for each $g \neq 1$ in $G$ (see [3, Proposition 1.2, page 80]). A natural generalization of a commutative Galois extension is the notion of a center Galois extension, that is, a noncommutative ring $B$ with a finite automorphism group $G$ and center $C$ is called a center Galois extension of $B^G$ with Galois group $G$ if $C$ is a Galois extension of $C^G$ with Galois group $G|C \cong G$. Ikehata (see [4, 5]) characterized a center Galois extension with a cyclic Galois group $G$ of prime order in terms of a skew polynomial ring. Then, the present authors generalized the Ikehata characterization to center Galois extensions with Galois group $G$ of any cyclic order [7] and to center Galois extensions with any finite Galois group $G$ [8]. The purpose of the present paper is to generalize the above characterization of a commutative Galois extension to a center Galois extension. We shall show that $B$ is a center Galois extension of $B^G$ if and only if the ideal of $B$ generated by \{c - g(c) \mid c \in C\} is $B$ for each $g \neq 1$ in $G$. A center Galois extension $B$ is also equivalent to each of the following statements:

(i) $B$ is a Galois central extension of $B^G$, that is, $B = B^G C$ which is a Galois extension of $B^G$.

(ii) $B$ is a Galois extension of $B^G$ with a Galois system \{b_i \in B, c_i \in C, i = 1, 2, \ldots, m\} for some integer $m$.

(iii) the ideal of the subring $B^G C$ generated by \{c - g(c) \mid c \in C\} is $B^G C$ for each $g \neq 1$ in $G$. 


2. Definitions and notations. Throughout this paper, $B$ will represent a ring with identity $1$, $G = \{g_1 = 1, g_2, \ldots, g_n\}$ an automorphism group of $B$ of order $n$ for some integer $n$, $C$ the center of $B$, $B^G$ the set of elements in $B$ fixed under each element in $G$, and $B \star G$ a skew group ring in which the multiplication is given by $gb = g(b)g$ for $b \in B$ and $g \in G$.

$B$ is called a $G$-Galois extension of $B^G$ if there exist elements $\{a_i, b_i \in B, i = 1, 2, \ldots, m\}$ for some integer $m$ such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_1$. Such a set $\{a_i, b_i\}$ is called a $G$-Galois system for $B$. $B$ is called a central Galois extension of $B^G$ if $C$ is a Galois algebra over $C^G$ with Galois group $G|_C \cong G$. $B$ is called a central extension of $B^G$ if $B = B^G C$, and $B$ is called a Galois central extension of $B^G$ if $B = B^G C$ is a Galois extension of $B^G$ with Galois group $G$.

Let $A$ be a subring of a ring $B$ with the same identity $1$. We denote $V_B(A)$ the commutator subring of $A$ in $B$. We call $B$ a separable extension of $A$ if there exist $\{a_i, b_i \in B, i = 1, 2, \ldots, m\}$ for some integer $m$ such that $\sum_{i=1}^{m} a_i b_i = 1$ and $\sum b_i a_i = \sum a_i b_i$ for all $b \in B$ where $\otimes$ is over $A$. $B$ is called $H$-separable extension of $A$ if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of $B$ as a $B$-bimodule.

$B$ is called centrally projective over $A$ if $B$ is a direct summand of a finite direct sum of $A$ as a $A$-bimodule.

3. The characterizations. In this section, we denote $J_j^C = \{c - g_j(c) \mid c \in C\}$. We shall show that $B$ is a center Galois extension of $B^G$ if and only if $B = BJ_j^C$, the ideal of $B$ generated by $J_j^C$, for each $g_j \neq 1$ in $G$. Some more characterizations of a center Galois extension $B$ are also given. We begin with a lemma.

**Lemma 3.1.** If $B = BJ_j^C$ for each $g_j \neq 1$ in $G$ (that is, $j \neq 1$), then

1. $B$ is a Galois extension of $B^G$ with Galois group $G$ and a Galois system $\{b_i \in B; c_i \in C, i = 1, 2, \ldots, m\}$ for some integer $m$.
2. $B$ is a centrally projective over $B^G$.
3. $B \star G$ is $H$-separable over $B$.

**Proof.** (1) Since $B = BJ_j^C$ for each $j \neq 1$, there exist $\{b_i^{(j)} \in B, c_i^{(j)} \in C, i = 1, 2, \ldots, m_j\}$ for some integer $m_j, j = 2, 3, \ldots, n$ such that $\sum_{i=1}^{m_j} b_i^{(j)} (c_i^{(j)} - g_j(c_i^{(j)})) = 1$. Therefore, $\sum_{i=1}^{m_j} b_i^{(j)} c_i^{(j)} = 1 + \sum_{i=1}^{m_j} b_i^{(j)} g_j(c_i^{(j)})$. Let $b_i^{(j)} = -\sum_{i=1}^{m_j} b_i^{(j)} g_j(c_i^{(j)})$ and $c_i^{(j)} = 1$. Then $\sum_{i=1}^{m_j} b_i^{(j)} c_i^{(j)} = 1$ and $\sum_{i=1}^{m_j+1} b_i^{(j)} g_j(c_i^{(j)}) = 0$. Let $b_{i_1, i_2, \ldots, i_m} = b_{i_2}^{(3)} b_{i_3}^{(3)} \cdots b_{i_n}^{(3)}$. Then

$$\sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2, i_3, \ldots, i_n} c_{i_2, i_3, \ldots, i_n} = \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(3)} b_{i_3}^{(3)} \cdots b_{i_n}^{(3)} c_{i_2}^{(3)} c_{i_3}^{(3)} \cdots c_{i_n}^{(3)}$$

$$= \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(3)} c_{i_2}^{(3)} b_{i_3}^{(3)} c_{i_3}^{(3)} \cdots b_{i_n}^{(3)} c_{i_n}^{(3)}$$

$$= \sum_{i_2=1}^{m_2+1} b_{i_2}^{(3)} c_{i_2}^{(3)} \sum_{i_3=1}^{m_3} b_{i_3}^{(3)} c_{i_3}^{(3)} \cdots \sum_{i_n=1}^{m_n} b_{i_n}^{(3)} c_{i_n}^{(3)} = 1$$

(3.1)
and for each $j \neq 1$

\[
\sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2,i_3,\ldots,i_n} g_j(c_{i_2,i_3,\ldots,i_n})
\]

\[
= \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} b_{i_3}^{(3)} \cdots b_{i_n}^{(n)} g_j(c_{i_2}^{(2)} c_{i_3}^{(3)} \cdots c_{i_n}^{(n)})
\]

\[
= \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} b_{i_3}^{(3)} \cdots b_{i_n}^{(n)} g_j(c_{i_2}^{(2)} c_{i_3}^{(3)} \cdots c_{i_n}^{(n)})
\]

\[
= \sum_{i_2=1}^{m_2+1} b_{i_2}^{(2)} g_j(c_{i_2}^{(2)}) \sum_{i_3=1}^{m_3+1} b_{i_3}^{(3)} g_j(c_{i_3}^{(3)}) \cdots \sum_{i_n=1}^{m_n+1} b_{i_n}^{(n)} g_j(c_{i_n}^{(n)}) = 0.
\]

Thus, \( \{b_{i_2,i_3,\ldots,i_n} \in B; c_{i_2,i_3,\ldots,i_n} \in C, i_j = 1,2,\ldots,m_j + 1 \text{ and } j = 2,3,\ldots,n \} \) is a Galois system for \( B \). This completes the proof of (1).

(2) By (1), \( B \) is a Galois extension of \( B^G \) with a Galois system \( \{b_i \in B, c_i \in C, i = 1,2,\ldots,m \} \) for some integer \( m \). Let \( f_i : B \to B^G \) given by \( f_i(b) = \sum_{j=1}^{n} g_j(c_i b) \) for all \( b \in B, i = 1,2,\ldots,m \). Then it is easy to check that \( f_i \) is a homomorphism as \( B^G \)-bimodule and \( b = \sum_{i=1}^{m} b_i c_i b = \sum_{j=1}^{m} \sum_{i=1}^{m} b_i g_j(c_i) g_j(b) = \sum_{i=1}^{m} b_i \sum_{j=1}^{m} g_j(c_i b) = \sum_{i=1}^{m} b_i f_i(b) \) for all \( b \in B \). Hence \( \{b_i, f_i, i = 1,2,\ldots,m \} \) is a dual bases for \( B \) as \( B^G \)-bimodule, and so \( B \) is finitely generated and projective as \( B^G \)-bimodule. Therefore, \( B \) is a direct summand of a finite direct sum of \( B^G \) as a \( B^G \)-bimodule. Thus \( B \) is centrally projective over \( B^G \).

(3) By (1), \( B \) is a Galois extension of \( B^G \) with Galois group \( G \). Hence \( B \ast G \cong \text{Hom}_{B^G}(B, B) \) [2, Theorem 1]. By (2), \( B \) is centrally projective over \( B^G \). Thus, \( B \ast G \cong \text{Hom}_{B^G}(B, B) \) is \( H \)-separable over \( B \) [6, Proposition 11].

(4) We first claim that \( V_{B^G}(C) = B \). Clearly, \( B \subset V_{B^G}(C) \). Let \( \sum_{j=1}^{n} b_j g_j \) in \( V_{B^G}(C) \) for some \( b_j \in B \). Then \( c(\sum_{j=1}^{n} b_j g_j) = (\sum_{j=1}^{n} b_j g_j) c \) for each \( c \in C \), so \( c b_j = b_j g_j(c) \), that is, \( b_j(c - g_j(c)) = 0 \) for each \( g_j \in G \) and \( c \in C \). Since \( B = B^G \) for each \( g_j \neq 1 \), there exist \( b_j^{(i)} \in B \) and \( c_i^{(j)} \in C, i = 1,2,\ldots,m \) such that \( \sum_{i=1}^{m} b_j^{(i)} (c_i^{(j)} - g_j(c_i^{(j)})) = 1 \). Hence \( b_j = \sum_{i=1}^{m} b_j^{(i)} (c_i^{(j)} - g_j(c_i^{(j)})) b_j = \sum_{i=1}^{m} b_j^{(i)} b_j (c_i^{(j)} - g_j(c_i^{(j)})) = 0 \) for each \( g_j \neq 1 \). This implies that \( \sum_{j=1}^{n} b_j g_j = b_1 \in B \). Hence \( V_{B^G}(C) \subset B \) and so \( V_{B^G}(C) = B \). Therefore, \( V_{B^G}(B) \subset V_{B^G}(C) = B \). Thus \( V_{B^G}(B) = V_B(B) = C \).

We now show some characterizations of a center Galois extension \( B \).

**Theorem 3.2.** The following statements are equivalent.

(1) \( B \) is a center Galois extension of \( B^G \).

(2) \( B = B^{G_j} \) for each \( g_j \neq 1 \) in \( G \).

(3) \( B \) is a Galois extension of \( B^G \) with a Galois system \( \{b_i \in B, c_i \in C, i = 1,2,\ldots,m \} \) for some integer \( m \).

(4) \( B \) is a Galois central extension of \( B^G \).

(5) \( B^G = B^{G_j} C^{G_j} \) for each \( g_j \neq 1 \) in \( G \).
\textbf{Proof.} (1)\implies(2). By hypothesis, \( C \) is a Galois extension of \( C^G \) with Galois group \( G|_C \cong G \). Hence \( C = C_{f_j}^{(C)} \) for each \( g_j \neq 1 \) in \( G \) [3, Proposition 1.2, page 80]. Thus, \( B = Bf_j^{(C)} \) for each \( g_j \neq 1 \) in \( G \).

(2)\implies(1). Since \( B = Bf_j^{(C)} \) for each \( g_j \neq 1 \) in \( G \), \( B \ast G \) is \( H \)-separable over \( B \) by Lemma 3.1(3) and \( V_{B \ast G}(B) = C \) by Lemma 3.1(4). Thus \( C \) is a Galois extension of \( C^G \) with Galois group \( G|_C \cong G \) by [1, Proposition 4].

(1)\implies(3). This is Lemma 3.1(1).

(3)\implies(1). Since \( B \) is Galois extension of \( B^G \) with a Galois system \( \{b_i \in B, c_i \in C, i = 1,2,\ldots,m\} \) for some integer \( m \), we have \( \sum_{i=1}^{m} b_i g_j(c_i) = \delta_{i,j} \). Hence \( \sum_{i=1}^{m} b_i (c_i - g_j(c_i)) = 1 \) for each \( g_j \neq 1 \) in \( G \). So for every \( b \in B \), \( b = \sum_{i=1}^{m} b b_i (c_i - g_j(c_i)) \in Bf_j^{(C)} \). Therefore, \( B = Bf_j^{(C)} \) for each \( g_j \neq 1 \) in \( G \). Thus, \( B \) is a center Galois extension of \( B^G \) by (2)\implies(1).

(1)\implies(4). Since \( C \) is a Galois algebra with Galois group \( G|_C \cong G \), \( B \) and \( B^G \) are Galois extensions of \( B^G \) with Galois group \( G|_{B^G C} \cong G \). Noting that \( B^G C \subset B \), we have \( B = B^G C \), that is, \( B \) is a central extension of \( B^G \). But \( B \) is a Galois extension of \( B^G \), so \( B \) is a Galois central extension of \( B^G \).

(4)\implies(1). By hypothesis, \( B = B^G C \) is a Galois extension of \( B^G \). Hence there exists a Galois system \( \{a_i; b_i \in B, i = 1,2,\ldots,m\} \) for some integer \( m \) such that \( \sum_{i=1}^{m} a_i g_j(b_i) = \delta_{i,j} \). But \( B = B^G C \), so \( a_i = \sum_{k=1}^{m} a_k^{(a_i)} b_k^{(a_i)} c_k^{(a_i)} \) and \( b_i = \sum_{l=1}^{m} b_l^{(b_i)} c_l^{(b_i)} \) for some \( a_k^{(a_i)}, b_l^{(b_i)} \) in \( B^G \) and \( c_k^{(a_i)}, c_l^{(b_i)} \) in \( C \), \( k = 1,2,\ldots,n_{a_i}, l = 1,2,\ldots,n_{b_i}, i = 1,2,\ldots,m \). Therefore,

\[ \delta_{i,j} = \sum_{i=1}^{m} a_i g_j(b_i) = \sum_{i=1}^{m} \sum_{k=1}^{n_{a_i}} b_k^{(a_i)} c_k^{(a_i)} g_j \left( \sum_{l=1}^{n_{b_i}} b_l^{(b_i)} c_l^{(b_i)} \right) = \sum_{i=1}^{m} \sum_{k=1}^{n_{a_i}} \sum_{l=1}^{n_{b_i}} \left( b_k^{(a_i)} c_k^{(a_i)} b_l^{(b_i)} c_l^{(b_i)} \right) g_j \left( c_l^{(b_i)} \right). \]

This shows that \( \{b_k^{(a_i)}, c_k^{(a_i)}, b_l^{(b_i)} \in B, c_k^{(a_i)}, c_l^{(b_i)} \in C, k = 1,2,\ldots,n_{a_i}, l = 1,2,\ldots,n_{b_i}, i = 1,2,\ldots,m \} \) is a Galois system for \( B \). Thus, \( B \) is a center Galois extension of \( B^G \) by (3)\implies(1).

(1)\implies(5). Since \( B \) is a center Galois extension of \( B^G \), \( B = Bf_j^{(C)} \) for each \( g_j \neq 1 \) in \( G \) by (1)\implies(2) and \( B = B^G C \) by (1)\implies(4). Thus, \( B^G C = B^G C f_j^{(C)} \) for each \( g_j \neq 1 \) in \( G \).

(5)\implies(1). Since \( B^G C = B^G C f_j^{(C)} \) for each \( g_j \neq 1 \) in \( G \), \( B = Bf_j^{(C)} \) for each \( g_j \neq 1 \) in \( G \). Thus, \( B \) is a center Galois extension of \( B^G \) by (2)\implies(1).

The characterization of a commutative Galois extension \( C \) in terms of the ideals generated by \( \{c - g(c) \mid c \in C\} \) for \( g \neq 1 \) in \( G \) is an immediate consequence of Theorem 3.2.

\textbf{Corollary 3.3.} A commutative ring \( C \) is a Galois extension of \( C^G \) if and only if \( C = C_{f_j}^{(C)} \), the ideal generated by \( \{c - g_j(c) \mid c \in C\} \) is \( C \) for each \( g_j \neq 1 \) in \( G \).

\textbf{Proof.} Let \( B = C \) in Theorem 3.2. Then, the corollary is an immediate consequence of Theorem 3.2(2).

By Theorem 3.2, we derive several characterizations of a Galois centreal extension \( B \).
\textbf{Corollary 3.4.} If \( B \) is a central extension of \( B^G \) (that is, \( B = B^G C \)), then the following statements are equivalent.
\begin{enumerate}
\item \( B \) is a Galois extension of \( B^G \).
\item \( B \) is a center Galois extension of \( B^G \).
\item \( B \ast G \) is \( H \)-separable over \( B \).
\item \( B = CJ_j(B) \) for each \( g_j \neq 1 \) in \( G \).
\item \( B = BJ_j(B) \) for each \( g_j \neq 1 \) in \( G \).
\end{enumerate}

\textbf{Proof.} (1)\( \iff \) (2). This is given by (1)\( \iff \) (4) in Theorem 3.2.
(2)\( \implies \) (3). This is Lemma 3.1(3).
(3)\( \implies \) (1). Since \( B \ast G \) is \( H \)-separable over \( B \), \( B \) is a Galois extension of \( B^G \) [1, Proposition 2].

Since \( B = B^G C \) by hypothesis, it is easy to see that \( J_j^{(B)} = B^G J_j^{(C)} \) for each \( g_j \) in \( G \). Thus, \( B = CJ_j(B) \), \( B = BJ_j(B) \), and \( B = BJ_j(C) \) are equivalent. This implies that (2)\( \iff \) (4)\( \iff \) (5) by Theorem 3.2(2).

We call a ring \( B \) the DeMeyer-Kanzaki Galois extension of \( B^G \) if \( B \) is an Azumaya \( C \)-algebra and \( B \) is a central Galois extension of \( B^G \) (for more about the DeMeyer-Kanzaki Galois extensions, see [2]). Clearly, the class of center Galois extensions is broader than the class of the DeMeyer-Kanzaki Galois extensions. We conclude the present paper with two examples. (1) The DeMeyer-Kanzaki Galois extension of \( B^G \) and (2) a center Galois extension of \( B^G \), but not the DeMeyer-Kanzaki Galois extension of \( B^G \).

\textbf{Example 3.5.} Let \( \mathbb{C} \) be the field of complex numbers, that is, \( \mathbb{C} = \mathbb{R} + \mathbb{R}\sqrt{-1} \) where \( \mathbb{R} \) is the field of real numbers, \( B = \mathbb{C}[i,j,k] \) the quaternion algebra over \( \mathbb{C} \), and \( G = \{1,g \mid g(c_1 + c_i i + c_j j + c_k k) = g(c_1) + g(c_i)i + g(c_j)j + g(c_k)k \text{ for each } b = c_1 + c_i i + c_j j + c_k k \in \mathbb{C}[i,j,k] \text{ and } g(u + v \sqrt{-1}) = u - v \sqrt{-1} \text{ for each } c = u + v \sqrt{-1} \in \mathbb{C} \}. \)
\begin{enumerate}
\item The center of \( B \) is \( \mathbb{C} \).
\item \( B \) is an Azumaya \( C \)-algebra.
\item \( \mathbb{C} \) is a Galois extension of \( C^G \) with Galois group \( G|\mathbb{C} \cong G \) and a Galois system \( \{a_1 = 1/\sqrt{2}, a_2 = (1/\sqrt{2}) \sqrt{-1}; b_1 = 1/\sqrt{2}, b_2 = -(1/\sqrt{2}) \sqrt{-1} \} \).
\item \( B \) is the DeMeyer-Kanzaki Galois extension of \( B^G \) by (2) and (3).
\item \( B^G = \mathbb{R}[i,j,k] \).
\item \( B = B^G \mathbb{C} \), so \( B \) is a central extension of \( B^G \).
\item \( J_j^{(C)} = \mathbb{R}\sqrt{-1} \).
\item \( B = BJ_j(C) \) since \( 1 = -\sqrt{-1}\sqrt{-1} = BJ_j(C) \).
\item \( J_j^{(B)} = \mathbb{R}\sqrt{-1} + \mathbb{R}\sqrt{-1}i + \mathbb{R}\sqrt{-1}j + \mathbb{R}\sqrt{-1}k \).
\item \( B = \mathbb{C}J_j^{(B)} \).
\end{enumerate}

\textbf{Example 3.6.} By replacing in Example 3.5 the field of complex numbers \( \mathbb{C} \) with the ring \( C = \mathbb{Z} \oplus \mathbb{Z} \) where \( \mathbb{Z} \) is the ring of integers, \( g(a,b) = (b,a) \) for all \( (a,b) \in C \), and \( G = \{1,g \mid g(c_1 + c_i i + c_j j + c_k k) = g(c_1) + g(c_i)i + g(c_j)j + g(c_k)k \text{ for each } b = c_1 + c_i i + c_j j + c_k k \in B = C[i,j,k] \} \). Then
\begin{enumerate}
\item The center of \( B \) is \( C \).
\item \( C \) is a Galois extension of \( C^G \) with Galois group \( G|\mathbb{C} \cong G \) and a Galois system \( \{a_1 = (1,0), a_2 = (0,1); b_1 = (1,0), b_2 = (0,1) \} \).
(3) \( B \) is not an Azumaya \( C \)-algebra (for \( \frac{1}{2} \not\in C \)), and so \( B \) is not the DeMeyer-Kanzaki Galois extension of \( B^G \).

(4) \( C^G = \{(a, a) \mid a \in \mathbb{Z}\} \cong \mathbb{Z} \).

(5) \( B^G = C^G[i, j, k] \).

(6) \( B = B^G C \), so \( B \) is a central extension of \( B^G \).

(7) \( J^{(C)}_G = \{(a, -a) \mid a \in \mathbb{Z}\} = \mathbb{Z}(1, -1) \).

(8) \( B = B J^{(C)}_G \) since \( 1 = (1, 1) = (1, -1)(1, -1) \in B J^{(C)}_G \).

(9) \( J^{(B)}_G = \mathbb{Z}(1, -1) + \mathbb{Z}(1, -1)i + \mathbb{Z}(1, -1)j + \mathbb{Z}(1, -1)k \).

(10) \( B = CJ^{(B)}_G \).

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