EXISTENCE AND UNIQUENESS THEOREMS FOR SOME FOURTH-ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

RUYUN MA

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Abstract. Let $G$ and $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be two functions satisfying Caratheodory conditions. This paper is concerned with the problems of existence and uniqueness of solutions for the nonlinear fourth-order ordinary differential equation

$$y'''' + \lambda y'' + ky + G(x, y, y', y'', y''') = f(x, y, y', y'', y''')$$

with one of a particular set of boundary conditions.

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1. Introduction. In this paper, we are concerned with the global solvability of the fourth-order ordinary differential equation

$$y'''' + \lambda y'' + ky + G(x, y, y', y'', y''') = f(x, y, y', y'', y''')$$

with one of the following sets of boundary conditions:

\begin{align*}
  y(0) &= y'(0) = y(1) = y''(1) = 0; \\
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  y(0) &= y'(0) = y'(1) = y''''(1) = 0.
\end{align*}

(Hereinafter, for simplicity, when we refer to (1.1) we will actually mean the fourth-order differential equation (1.1) along with one of the boundary conditions given in (1.2) through (1.7).) Those boundary value problems govern the equilibrium states of a beam-column. The source of nonlinearity comes from a nonlinear lateral constraint (foundation). The equilibrium equation is formulated as a fourth-order nonlinear differential equation. Different boundary conditions are corresponding to various ways in which the ends of the beam may be supported.

Very recently, Elgindi and Guan [3] studied these boundary value problems when $G$ in (1.1) is independent of $y'''$, and $f$ in (1.1) is independent of $y, y', y''$, and $y'''$, that is, $G(x, y, y', y'', y''') = G(x, y, y', y'')$ and $f(x, y, y', y'', y''') = f(x)$. It turns
out that the dependence of $G$ and $f$ in (1.1) on the third-order derivative $y'''$ of $y$ causes a fundamental difference between the problem studied in this paper and the corresponding problems studied in [3], where $G$ and $f$ are independent of $y'''$. The conditions on $G(x, y, y', y'')$ for the existence of a solution to the boundary value problems (1.1), as given in [3], are some homogeneous conditions and sign conditions with respect to $y$. Now when $G$, in (1.1), is not independent of $y'''$, these conditions on $G$ which are related to $y$ do not give the necessary, a priori, estimates to obtain existence of a solution to the boundary value problems (1.1). It is the purpose of this paper to show that necessary, a priori, estimates can be obtained to prove existence of a solution to the boundary value problems (1.1) when one imposes conditions on $G$ and $f$ they are related to $y'''$. We remark that the existence and uniqueness theorems obtained in this paper for the boundary value problems (1.1), when particularized to the case when $G$ in (1.1) is independent of $y'''$ and $f(x, y, y', y'', y''') = f(x)$ gives new existence theorems for the problems studied in [3]. We would like to refer the reader to [1, 2, 4, 5, 6, 7, 8] and references therein for related works on fourth-order boundary value problems.

The proof of the existence of solutions is based upon a corollary of Leray-Schauder fixed point theorem, which we state here as the following lemma.

**Lemma 1.1.** Let $B$ be a banach space and $K : B → B$ be a compact operator. Suppose that there exists a priori bound $m > 0$ such that every solution of $y - tKy = 0$, for $t ∈ [0, 1]$, satisfies $∥y∥ ≤ m$. Then $K$ has a fixed point $y$ with $∥y∥ ≤ m$.

### 2. Assumptions and preliminary results

**Definition 2.1.** A function $u : [0, 1] × \mathbb{R}^k → \mathbb{R}$ is said to satisfy Corotheodory’s condition for $L^q(0, 1)$ if the following conditions are satisfied:

(i) for a.e. $x ∈ [0, 1]$, the function $f(x, ·)$ is continuous;
(ii) for every $y ∈ \mathbb{R}^k$, the function $f(·, y)$ is measurable;
(iii) for every $r > 0$, there is $g_r ∈ L^q(0, 1)$ such that for a.e. $x ∈ [0, 1]$, $|u(x, y)| ≤ g_r(x)$ whenever $∥y∥ ≤ r$.

Throughout the rest of the paper, we use the following notation:

\[
W^k = \left\{ y : [0, 1] → \mathbb{R} | y^{(j)} ∈ AC[0, 1], j = 0, 1, ... , k - 1 \text{ and } y^{(k)} ∈ L^2(0, 1) \right\}, \tag{2.1}
\]

\[
∥y∥^2_k = \sum_{j=0}^{k} ∥y^{(j)}∥^2_k, \quad y ∈ W^k,
\]

\[
D(L_i) = \{ y ∈ W^k | y \text{ satisfies the } i\text{th boundary conditions (1.i)}, i = 2, ... , 7, \quad L_j : D(L_i) → L^2(0, 1) \text{ is defined by} \n\]

\[
L_j(y) = y^{''''}.
\]

We make the following assumptions.

(H1) Let $f : [0, 1] × \mathbb{R}^4 → \mathbb{R}$ satisfy Corotheodory’s condition for $L^2(0, 1)$ and there exists $\bar{f} ∈ L^2(0, 1)$ such that

\[
|f(x, y, z, u, v)| ≤ \bar{f}(x) \quad \text{for all } (x, y, z, u, v) ∈ [0, 1] × \mathbb{R}^4. \tag{2.3}
\]
**Lemma 2.2.** If \( u \in C^1[0,1] \), and \( u(0) = 0 \), then \( \|u\|_{L^2}^2 \leq (4/\pi^2)\|u'\|_{L^2}^2. \)

**Lemma 2.3.** If \( u \in C^1[0,1] \), and \( u(0) = u(1) \), then \( \|u\|_{L^2}^2 \leq (1/\pi^2)\|u'\|_{L^2}^2. \)

**Lemma 2.4.** Let \( M_\eta = \max\{\eta, 1 - \eta\} \), \( 0 \leq \eta \leq 1 \). If \( u \in C^1[0,1] \), and \( u(\eta) = 0 \), then
\[
\|u\|_{L^2}^2 \leq \left( \frac{4}{\pi^2} \right) M_\eta \|u'\|_{L^2}^2.
\]

Lemmas 2.2 and 2.3 are direct consequences of Wirtinger's inequalities, see [1]. Lemma 2.4 can be easily deduced from Lemmas 2.2 and 2.3.

**Lemma 2.5.** For each \( L_j \), \( j = 2, \ldots, 7 \), the following are true:

(A) \( L_j \), as an operator on \( L^2 \), is densely defined and self-adjoint;

(B) \( c_j \|y\|_{L^2} \leq \|y''\|_{L^2} \leq d_j \|y'''\|_{L^2} \) for \( y \in D(L_j) \), where \( c_2 = c_5 = \pi^2 \), \( c_3 = c_7 = \pi^2/2 \), \( c_4 = c_6 = \pi^2/4 \); \( d_2 = 1/\pi \), \( d_3 = d_4 = d_5 = d_6 = d_7 = 2/\pi \);

(C) for any \( y \in D(L_j) \), \( L_j y = 0 \) if and only if \( y = 0 \);

(D) there exists unique \( \psi_j : L^2(0,1) \to W^4 \) such that \( L_j(\psi_j(h)) = h \) for any \( h \in L^2(0,1) \) and \( \psi_j : L^2(0,1) \to W^4 \) is bounded;

(E) \( K_j : L^2 \to W^3 \) defined by \( K_j = i \circ \psi_j \), where \( i : D(L_j) \to W^3 \) denotes the identity map, is compact.

The proof of (A)-(E) are direct and therefore omitted. For some of the estimates in (B), one needs to use Lemmas 2.2, 2.3, and 2.4.

### 3. Existence of solutions.

In this section, we consider the solvability of the six boundary value problems consisting of the differential equation (1.1) in the following theorem.

**Theorem 3.1.** Under the Assumptions (H1) and (H2), the boundary value problem consisting of (1.1), and \((1,j),\ j = 2, \ldots, 7\), has at least one solution.

**Proof.** The boundary value problem (1.1) can be written as
\[
y' = K_j y,
\]
where
\[
K_j y = K_j [\lambda y'' + k y + G(x, y, y', y'', y''') - f(x, y, y', y'', y''')].
\]
$K_j : W^3 \to W^3$ is compact, and $\mathcal{K}_j$ is as in Lemma 2.5. We prove the existence of solution of (3.1) by verifying the conditions of Lemma 1.1.

Assume that the solutions of $y \cdot tK_jy = 0$ are not uniformly bounded with respect to $t \in [0, 1]$. Then there exist sequence $\{t_n\} \subset (0, 1)$ and $y_n \in W^3$ such that

$$y_n = t_n K_j y_n$$

and $\|y_n\|_3 \to \infty$ as $n \to \infty$.

From (3.3), it follows that each $y_n$ satisfies

$$y'''_n + t_n \lambda y''_n + t_n k y_n + t_n G(x, y_n, y'_n, y''_n, y'''_n) = t_n f(x, y_n, y'_n, y''_n, y'''_n)$$

with $y_n \in D(L_j)$, which in turn implies (upon multiplying both sides of the equation by $y''_n$, integrating by parts and using the boundary conditions)

$$-\|y''_n\|_{L^2}^2 + t_n \lambda \|y''_n\|_{L^2}^2 + t_n k \int_0^1 y_n y''_n \, dx + t_n \int_0^1 G(x, y_n, y'_n, y''_n, y'''_n) y''_n \, dx$$

$$= t_n \int_0^1 f(x, y_n, y'_n, y''_n, y'''_n) y''_n \, dx.$$  

Set $z_n = y_n/\|y_n\|_3$, then $\{z_n\} \subset W^3$ is a bounded sequence, and since a bounded set of $W^3$ is weakly relatively compact, it follows that there exists a subsequence of $\{z_n\}$, that converges weakly in $W^3$. By the fact that the embedding $i_0 : D(L) \subset W^3 \to C^2[0, 1]$ is compact, it follows that there exists a subsequence of $\{z_n\}$, which we recall $\{z_n\}$ again, that converges strongly in $C^2[0, 1]$ to some $z_0 \in C^2[0, 1]$.

From (3.5) and Assumption (H2), we obtain

$$-t_n \int_0^1 g(y''_n) y'_n \, dx = -\|y''_n\|_{L^2}^2 + t_n k \int_0^1 y_n y''_n \, dx + t_n \lambda \|y''_n\|_{L^2}^2$$

$$+ t_n \int_0^1 h(x, y_n, y'_n, y''_n, y'''_n) y''_n \, dx$$

$$-t_n \int_0^1 f(x, y_n, y'_n, y''_n, y'''_n) y''_n \, dx$$

$$\leq t_n |k| \|y''_n\|_{L^2} \|y_n\|_{L^2} + t_n |\lambda| \|y''_n\|_{L^2}^2$$

$$-t_n \int_0^1 f(x, y_n, y'_n, y''_n, y'''_n) y''_n \, dx$$

$$\leq t_n (c_j^{-1} |k| + |\lambda|) \|y''_n\|_{L^2}^2 + t_n \|f\|_{L^2} \|y''_n\|_{L^2}.$$  

Using (3.6) and homogeneity of $g$, we obtain

$$0 \leq -t_n \int_0^1 g(y''_n) z''_n \, dx \leq \frac{(c_j^{-1} |k| + |\lambda|) \|y''_n\|_{L^2}^2}{\|y_n\|_3^{p+1}} + \frac{\|f\|_{L^2} \|y''_n\|_{L^2}}{\|y_n\|_3} \to 0$$  

as $n \to \infty$. Since $g$ is continuous, it follows from (3.7) that $\int_0^1 g(z''_n) z''_n \, dx = 0$ which, in view of (H2) part (b), implies that $z''_n = 0$. This together with the boundary conditions (1.2) through (1.7) imply that $z_0 = 0$. Thus $z_n \to 0$ in $C^2[0, 1]$. 

On the other hand, from (3.5) we have
\[
\frac{||y'''||^2_{L^2}}{||y'||^2_3} = t_n \lambda \frac{||y''||^2_{L^2}}{||y'||^2_3} + t_n k \int_0^1 \frac{y_n y'''}{||y'||^2_3} dx \\
+ t_n \int_0^1 G(x, y_n, y', y'', y''') \frac{y'''}{||y'||^2_3} dx \\
- t_n \int_0^1 f(x, y_n, y', y'', y''') \frac{y'''}{||y'||^2_3} dx
\]  
(3.8)
which implies that (by the fact that \(z_n \to 0\) in \(C^2[0,1]\))
\[
\frac{||y'''||^2_{L^2}}{||y'||^2_3} \to 0. 
\]  
(3.9)
However, from part (B) of Lemma 2.5 and Lemmas 2.2 and 2.3, we know that
\[
||y'||^2_{L^2} \leq ||y''||^2_{L^2} \leq ||y''''||^2_{L^2} \quad \text{for } y \in D(L) 
\]  
(3.10)
and moreover, we have
\[
||y||^2_3 = ||y'||^2_{L^2} + ||y''||^2_{L^2} + ||y'''||^2_{L^2} + ||y''''||^2_{L^2} \leq 4||y''''||^2_{L^2},
\]  
(3.11)
and this contradicts (3.9). This completes the proof.

4. Uniqueness. Assume that \(G(x, y, y', y'', y''')\) and \(f(x, y, y', y'', y''')\) satisfy the condition (H3) for all \(y, z \in W^3\),
\[
\int_0^1 \left\{ \left[ G(x, y, y', y'', y''') - f(x, y, y', y'', y''') \right] \\
- \left[ G(x, z, z', z'', z''') - f(x, z, z', z'', z''') \right] \right\} (y'''' - z''') \, dx < 0.
\]  
(4.1)
We obtain the following result on the uniqueness of the solution.

**Theorem 4.1.** Assume (H3), the solution of the boundary value problem (1.1) has at most one solution, provided that \(|\lambda| d_j^2 + |k| c_j^{-1} d_j^2 \leq 1\).

**Proof.** Let \(y\) and \(z\) be two solutions of the boundary value problem. Set \(w = y - z\) and assume that \(w'''' \neq 0\), \(w\) satisfies the equation
\[
w'''' + \lambda w'''' + kw + \left[ G(x, y, y', y'', y''') - f(x, y, y', y'', y''') \right] \\
- \left[ G(x, z, z', z'', z''') - f(x, z, z', z'', z''') \right] = 0
\]  
(4.2)
and the boundary condition (1.2) through (1.7). Let \(A = ||w''''||_{L^2}\) and \(B = ||w''''||_{L^2}\). Upon multiplying (4.2) by \(w''''\) and integrating by parts, using the boundary conditions, Holder’s inequality and (H3), we obtain
\[
-A^2 + \lambda B^2 + |k| c_j^{-1} B^2 > 0.
\]  
(4.3)
If \(|\lambda|d_j^2 + |k|c_j^{-1}d_j^2 \leq 1\), we have
\[
-A^2 + \lambda B^2 + |k|c_j^{-1}B^2 \leq -A^2 + |\lambda|d_j^2 A^2 + |k|c_j^{-1}d_j^2 A^2
\leq A^2 (-1 + |\lambda|d_j^2 + |k|c_j^{-1}d_j^2) \leq 0,
\]
(4.4)
where
\[
d_j = \begin{cases} 
\frac{1}{\pi}, & \text{for } j = 2, \\
\frac{2}{\pi}, & \text{for } j = 3, 4, 5, 6, 7,
\end{cases}
\]
(4.5)
(see part (B) of Lemma 2.5). (4.4) contradicts the inequality (4.3).

Thus \(w'' = 0\). This together with one of the boundary conditions (1.2), (1.3), (1.4), (1.5), (1.6), (1.7) imply that \(w = 0\). This proves the theorem. \(\Box\)

**Remark 4.2.** It is quite clear that most of the functions \(G\) which are of interest physically satisfy our Assumptions (H2) parts (a), (b), and (c), and (H3). For example, \(G(x,y,z,u,v) = -u^3\) satisfies all these assumptions. More generally, if we assume \(c_j \in L^2(0,1), j = 3, 5, \ldots, 2n + 1\), are functions satisfying \(c_{2n+1}(x) < 0\) and \(c_j(x) \leq 0\) for \(j = 3, 5, \ldots, 2n - 1\), then the functions \(G(x,y,z,u,v) = c_3(x)u^3 + c_5(x)u^5 + \cdots + c_{2n+1}(x)u^{2n+1}\) and \(g(u) = c_{2n+1}(x)u^{2n+1}\) satisfy our Assumptions (H2) and (H3).

**References**


Ma: Department of Mathematics, Northwest Normal University, Lanzhou 730070, Gansu, China

E-mail address: mary@nwnu.edu.cn