UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT. We study some integral operators and determine conditions for the univalence of these integral operators.

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1. Introduction. Let $A$ be the class of the functions $f$ which are analytic in the unit disc $U = \{ z \in \mathbb{C}; |z| < 1 \}$ and $f(0) = f'(0) = 1 = 0$.

We denote by $S$ the class of the functions $f \in A$ which are univalent in $U$.

Many authors studied the problem of integral operators which preserve the class $S$. In this sense an important result is due to Pfaltzgraff [4].

**Theorem 1.1** [4]. If $f$ is univalent in $U$, $\alpha$ a complex number and $|\alpha| \leq 1/4$, then the function

$$G_\alpha(z) = \int_0^z \left[f'(\xi)\right]^\alpha d\xi$$

(1.1)

is univalent in $U$.

**Theorem 1.2** [3]. If the function $g \in S$ and $\alpha$ is a complex number, $|\alpha| \leq 1/(4n)$, then the function defined by

$$G_{\alpha,n}(z) = \int_0^z \left[g'(u^n)\right]^\alpha du$$

(1.2)

is univalent in $U$ for all positive integer $n$.

2. Preliminary results. We need the following theorems.

**Theorem 2.1** [2]. Let $\alpha$ be a complex number, $\Re \alpha > 0$ and $f \in A$. If

$$\frac{1 - |z|^{2\Re \alpha}}{\Re \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1$$

(2.1)

for all $z \in U$, then for any complex number $\beta$, $\Re \beta \geq \Re \alpha$ the function

$$F_\beta(z) = \left[ \beta \int_0^z u^{\beta-1} f'(u) du \right]^{1/\beta}$$

(2.2)

is in the class $S$. 

**Theorem 2.2** [1]. If the function $g$ is regular in $U$ and $|g(z)| < 1$ in $U$, then for all $\xi \in U$ and $z \in U$ the following inequalities hold:

$$\left| \frac{g(\xi) - g(z)}{1 - g(z)g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \xi \bar{z}} \right|, \quad (2.3)$$

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}, \quad (2.4)$$

the equalities hold only in the case $g(z) = \epsilon (z + u)/(1 + \bar{u}z)$, where $|\epsilon| = 1$ and $|u| < 1$.

**Remark 2.3** [1]. For $z = 0$, from inequality (2.3)

$$\left| \frac{g(\xi) - g(0)}{1 - g(0)g(\xi)} \right| \leq |\xi| \quad (2.5)$$

and, hence

$$|g(\xi)| \leq \left| \frac{\xi + |g(0)|}{1 + |g(0)||\xi|} \right|. \quad (2.6)$$

Considering $g(0) = a$ and $\xi = z$,

$$|g(z)| \leq \left| \frac{z + |a|}{1 + |a||z|} \right|. \quad (2.7)$$

for all $z \in U$.

**Schwarz Lemma** [1]. If the function $g$ is regular in $U$, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold:

$$|g(z)| \leq |z| \quad (2.8)$$

for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (2.8) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3. Main results

**Theorem 3.1.** Let $\alpha$, $\gamma$ be complex numbers, $\text{Re} \alpha = a > 0$ and $g \in A$.

If

$$\left| \frac{g''(z)}{g'(z)} \right| \leq \frac{1}{n} \quad (3.1)$$

for all $z \in U$ and

$$|\gamma| \leq \frac{n + 2a}{2} \left( \frac{n + 2a}{n} \right)^{n/2a}, \quad (3.2)$$

then for any complex number $\beta$, $\text{Re} \beta \geq a$, the function

$$G_{\beta,\gamma,n}(z) = \left\{ \beta \int_0^z u^{\beta-1} [g'(u^n)] y^\gamma du \right\}^{1/\beta} \quad (3.3)$$

is in the class $S$ for all $n \in N^* - \{1\}$.
Proof. Let us consider the function

\[ f(z) = \int_0^z [g'(u^n)]^y \, du. \] (3.4)

The function

\[ p(z) = \frac{1}{|y|} f''(z), \] (3.5)

where the constant \(|y|\) satisfies the inequality (3.2), is regular in \(U\).

From (3.4) and (3.5), we obtain

\[ p(z) = \frac{y}{|y|} \left[ n z^{n-1} \frac{g''(z^n)}{g'(z^n)} \right]. \] (3.6)

Using (3.1) and (3.6) we obtain

\[ |p(z)| < 1 \] (3.7)

for all \(z \in U\). For \(z = 0\) we have \(p(0) = 0\).

From (3.6) and Schwarz lemma it results that

\[ \frac{1}{|y|} \left| \frac{f''(z)}{f'(z)} \right| \leq |z|^{n-1} \leq |z| \] (3.8)

for all \(z \in U\), and hence

\[ \left( \frac{1 - |z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq |y| \left( \frac{1 - |z|^{2a}}{a} \right) |z|^n. \] (3.9)

Let us consider \(Q : [0, 1] \rightarrow R, Q(x) = ((1 - x^{2a})/a)x^n, x = |z|\). We have

\[ Q(x) \leq \frac{2}{n+2a} \left( \frac{n}{n+2a} \right)^{n/2a} \] (3.10)

for all \(x \in [0, 1]\). From (3.2), (3.9), and (3.10) we obtain

\[ \left( \frac{1 - |z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \] (3.11)

for all \(z \in U\). Then, from (3.11) and Theorem 2.1 it follows that the function \(G_{\beta, \gamma, n}\) is in the class \(S\).

\[ \square \]

Theorem 3.2. Let \(\alpha, y\) be complex numbers, \(\text{Re } \alpha = b > 0\) and the function \(g \in A, g(z) = z + a_2 z^2 + \ldots\) If

\[ \left| \frac{g''(z)}{g'(z)} \right| < 1 \] (3.12)

for all \(z \in U\) and the constant \(|y|\) satisfies the condition

\[ |y| \leq \frac{1}{\max_{|z| \leq 1} \left[ ((1 - |z|^{2b})/b)|z|/((|z| + 2|a_2|)/(1 + 2|a_2| |z|)) \right]} \] (3.13)
then for any complex number \( \beta \), \( \text{Re} \beta \geq b \) the function

\[
G_{\beta,\gamma}(z) = \left\{ \beta \int_0^z u^{\beta-1} [g'(u)]^\gamma du \right\}^{1/\beta}
\]  

(3.14)
is in the class \( S \).

**Proof.** Let us consider the function

\[
f(z) = \int_0^z [g'(u)]^\gamma du.
\]

(3.15)
The function

\[
h(z) = \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)},
\]

(3.16)
where the constant \( |\gamma| \) satisfies the inequality (3.13), is regular in \( U \).

From (3.15) and (3.16) we have

\[
h(z) = \frac{\gamma}{|\gamma|} \frac{g''(z)}{g'(z)}.
\]

(3.17)
Using (3.12) and (3.17) we obtain

\[
|h(z)| < 1,
\]

(3.18)
for all \( z \in U \) and \( |h(0)| = 2|a_2| \).

Remark 2.3 applied to the function \( h \) gives

\[
\frac{1}{|\gamma|} \left| \frac{f''(z)}{f'(z)} \right| \leq \frac{|z| + 2|a_2|}{1 + 2|a_2||z|}
\]

(3.19)
for all \( z \in U \).

From (3.19) we obtain

\[
\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \frac{1 - |z|^{2b}}{b} \frac{|z| + 2|a_2|}{1 + 2|a_2||z|}
\]

(3.20)
for all \( z \in U \). Hence, we have

\[
\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \max_{|z| \leq 1} \left[ \frac{1 - |z|^{2b}}{b} \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \right].
\]

(3.21)
From (3.13) and (3.21) we obtain

\[
\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1
\]

(3.22)
for all \( z \in U \). From Theorem 2.1, it follows that the function \( G_{\beta,\gamma} \) defined by (3.14) is in the class \( S \).
REFERENCES


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