A NOTE ON $M$-IDEALS IN CERTAIN ALGEBRAS OF OPERATORS

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(Received 7 October 1998)

ABSTRACT. Let $X = (\sum_{n=1}^{\infty} \ell_1^n)^p$, $p > 1$. In this paper, we investigate $M$-ideals which are also ideals in $L(X)$, the algebra of all bounded linear operators on $X$. We show that $K(X)$, the ideal of compact operators on $X$ is the only proper closed ideal in $L(X)$ which is both an ideal and an $M$-ideal in $L(X)$.

Keywords and phrases. Compact operators, ideal, $M$-ideal.

2000 Mathematics Subject Classification. Primary 46A32.

1. Introduction. Since Alfsen and Effros [1, 2] introduced the notion of an $M$-ideal in a Banach space, many authors have studied $M$-ideals in algebras of operators. An interesting problem has been characterizing and finding those Banach spaces $X$ for which $K(X)$, the space of all compact linear operators on $X$, is an $M$-ideal in $L(X)$, the space of all continuous linear operators on $X$ [4, 8, 9, 11, 12].

It is known that if $X$ is a Hilbert space, $\ell_p$ $(1 < p < \infty)$ or $c_0$, then $K(X)$ is an $M$-ideal in $L(X)$ [6, 8, 12] while $K(\ell_1)$ and $K(\ell_\infty)$ are not $M$-ideals in the corresponding spaces of operators [12]. Smith and Ward [12] proved that $M$-ideals in a complex Banach algebra with identity are subalgebras and that they are two-sided algebraic ideals if the algebra is commutative. They also proved that $M$-ideals in a $C^*$-algebra are exactly the two-sided ideals [12]. Later, Cho and Johnson [5] proved that if $X$ is a uniformly convex Banach space, then every $M$-ideal in $L(X)$ is a left ideal, and if $X^*$ is also uniformly convex, then every $M$-ideal in $L(X)$ is a two-sided ideal in $L(X)$.

Flinn [7], and Smith and Ward [13] proved that $K(\ell_p)$ is the only nontrivial $M$-ideal in $L(\ell_p)$ for $1 < p < \infty$. Kalton and Werner [10] proved that if $1 < p$, $q < \infty$, $X = (\sum_{n=1}^{\infty} \ell_1^n)^p$ with complex scalars, then $K(X)$ is the only nontrivial $M$-ideal in $L(X)$. In their proof of this fact, Kalton and Werner [13] used the uniform convexity of $X$ and $X^*$. In this case, $M$-ideals in $L(X)$ are two-sided closed ideals in $L(X)$ [5].

In this paper, we investigate $M$-ideals which are also ideals in $L(X)$ for $X = (\sum_{n=1}^{\infty} \ell_1^n)^p$, $1 < p < \infty$. In our case, neither $X$ nor $X^*$ is uniformly convex. Therefore, no relationships between $M$-ideals and algebraic ideals in $L(X)$ seem to be known. But still we can use Kalton and Werner’s proof in [10] without using uniform convexity of $X$ and $X^*$ to prove that $K(X)$ is the only nontrivial $M$-ideal in $L(X)$ which is also a closed ideal in $L(X)$ (Theorem 3.3). By duality we have the same conclusion for the space $(\sum_{n=1}^{\infty} \ell_\infty^n)^p$, $1 < p < \infty$.

2. Preliminaries. A closed subspace $J$ of a Banach space $X$ is said to be an $L$-summand (respectively, $M$-summand) if there exists a closed subspace $J'$ of $X$
such that $X$ is an algebraic direct sum $X = J \oplus J'$ and satisfies a norm condition $\|j + j'\| = \|j\| + \|j'\|$ (respectively, $\|j + j'\| = \max\{\|j\|, \|j'\|\}$) for all $j \in J$ and $j' \in J'$.

In this case, we write $X = J \oplus_1 J'$ (respectively, $X = J \oplus_\infty J'$) and the projection $P$ on $X$ with rang $J$ is called an $L$-projection (respectively, an $M$-projection). A closed subspace $J$ of a Banach space $X$ is an $M$-ideal in $X$ if the annihilator $J^\perp$ of $J$ is an $L$-summand in $X^*$.

Let $A$ be a complex Banach algebra with identity $e$. The state space $S$ of $A$ is defined to be $\{\phi \in A^* : \phi(e) = \|\phi\| = 1\}$. An element $h \in A$ is said to be Hermitian if $\|e^{it\lambda}h\| = 1$ for all real number $\lambda$. Equivalently, $h$ is Hermitian if and only if $\phi(h)$ is real for every $\phi \in S$ [3, page 46].

In what follows, $Z$ always denote a complex Banach space $(\sum_{n=1}^\infty \ell^p_n)_p$, the $\ell^p$-sum of $\ell^p_n$’s for $1 < p < \infty$. For each $n$, let $\{e_{nt}\}_{t=1}^n$ be the standard basis of $\ell^p_n$. Then these bases string together to form the standard basis $\{e_n\}_{n=1}^\infty$ of $Z$ and each $T \in L(Z)$ has a matrix representation with respect to $\{e_n\}_{n=1}^\infty$. If $T \in L(Z)$ has the matrix whose $(i, j)$-entry is $t_{ij}$, then we can write $T = \sum_{i,j=1}^n t_{ij}e_j \otimes e_i$, where $e_j \otimes e_i$ is the rank 1 map sending $e_j$ to $e_i$. Observe that $T(e_j)$ forms the $j$th column vector of the matrix of $T$ and $\|Te_j\| \leq \|T\|$ for all $j = 1, 2, \ldots$. If the matrix of $T$ has at most one nonzero entry in each row and column, then $\|T\|$ is the $l^\infty$-norm of the sequence of nonzero entries.

A number of facts which hold in $L(\ell_p), 1 < p < \infty$, still hold in $L(Z)$. If the matrix of $T \in L(Z)$ is a diagonal matrix $(t_{ij})$ with real diagonal entries, then for each real $\lambda$ the matrix of $e^{i\lambda T}$ is also a diagonal matrix with diagonal entries $e^{i\lambda t_{ii}}$. Thus $T \in L(Z)$ is Hermitian if the matrix $T$ is a diagonal matrix with real entries.

Flinn [7] proved that if $M$ is an $M$-ideal in $L(\ell_p), 1 < p < \infty$ and $h$ is a Hermitian element in $L(\ell_p)$ with $h^2 = I$, then $hM \subseteq M$ and $Mh \subseteq M$. From this he proved that if $h$ is any diagonal matrix with real entries, then $hM \subseteq M$ and $Mh \subseteq M$. His proof is valid for $Z$ in place of $\ell_p$. Thus we have the following.

**Lemma 2.1.** If $M$ is an $M$-ideal in $L(Z)$ and $h \in L(Z)$ is a diagonal matrix with real entries, then $hM \subseteq M$ and $Mh \subseteq M$.

The $M$-ideal structure of $L(X)$ for $X = (\sum_{n=1}^\infty \ell^p_n)_p, 1 < p, q < \infty$ was studied by Kalton and Werner [10]. Some of their proofs for $X$ are still good for $Z$. One of them is the following.

**Lemma 2.2.** There is a constant $C$ such that, whenever $(k_n)$ is a sequence of positive integers with $\limsup k_n = \infty$, then $(\sum_{n=1}^\infty \ell^p_{k_n})_p$ is $C$-isomorphic to $(\sum_{n=1}^\infty \ell^n_1)_p$.

**Proof.** See proof of Lemma 3.1 of [10].

We recall that a Banach space $X$ is $C$-isomorphic to a Banach space $Y$ if there exists an isomorphism $T$ form $X$ onto $Y$ such that

$$1/C \|x\| \leq \|Tx\| \leq C\|x\| \quad (2.1)$$

for every $x \in X$. We use the following lemma which is due to Kalton and Werner [10].

**Lemma 2.3** [10]. Let $X$ be a Banach space, $\mathcal{F} \subset L(X)$ be a two-sided ideal, and $P$ a projection onto a complemented subspace $E$ of $X$ which is isomorphic to $X$.

(a) If $P \in \mathcal{F}$, then $\mathcal{F} = L(X)$. 


(b) If $E$ is $C$-isomorphic with $X$ and $\mathcal{T}$ contains an operator $T$ with $\|T - P\| < (C\|P\|^{-1})$, then $\mathcal{T} = L(X)$.

3. $M$-ideals in $L((\sum_{n=1}^{\infty} \ell_1^n)_p)$. A matrix carpentry used by Flinn [7] to characterize the $M$-ideal structure in $L(\ell_p)$ can be used to some extent in our case $Z = (\sum_{n=1}^{\infty} \ell_1^n)_p$. The proof of the following lemma is really a minor modification of Flinn’s proof in [7].

**Lemma 3.1.** If $M$ is a nontrivial $M$-ideal in $L(Z)$, then $K(Z) \subseteq M$.

**Sketch of the proof.** Let us call two positive integers $i$ and $j$ are in the same block if $n(n + 1)/2 < i, j \leq (n + 1)(n + 2)/2$ for some $n$. Using Lemma 2.1, we can follow Flinn’s proof of the second corollary to Lemma 1 in [7]. The only modification is the following: to prove $2^{1/q} < |t_{pl} + t_{kl}| \leq 2^{1/q}$, we consider two cases. If $p$ and $k$ are in a different block, Flinn’s proof just run through. If $p$ and $k$ are in the same block, then $2^{1/q} < |t_{pl} + t_{kl}| \leq \|T(e_{1})\| \leq 2^{1/q}$.

The proof of the following lemma is contained in the proof of Theorem 3.3 in [10].

**Lemma 3.2.** If $\mathcal{T}$ is a closed ideal strictly containing $K(Z)$ then $\mathcal{T}$ contains all the operators which factor through $\ell_p$.

The proof of the following theorem is a modification of that of Kalton and Werner [10]. Here we can go around the use of uniform convexity.

**Theorem 3.3.** If $\mathcal{T}$ is a closed ideal and also an $M$-ideal in $L(Z)$ strictly containing $K(Z)$, then $\mathcal{T} = L(Z)$.

**Proof.** We recall that the standard basis $\{e_{nl}\}^n_{l=1}$ of $\ell_1^n$ string together to form the standard basis $\{e_n\}^\infty_{n=1}$ of $Z$. If $\{e'_{nl}\}^n_{l=1}$ is the standard basis of $\ell_p$, then the map $e_n - e'_n$ gives a contraction from $Z$ to $\ell_p$. Since $E = \overline{\text{span}}\{e_{nl}\}^\infty_{n=1}$ is isometric to $\ell_p$, there exists a norm one operator $A$ from $Z$ to $E$ carrying $e_n$ to $e_{nl}$ via $e'_n$. Thus $A$ factors through $\ell_p$. By Lemma 3.2, $A \in \mathcal{T}$.

Since $\mathcal{T}$ is also an $M$-ideal, by Proposition 2.3 in [14], there exists a net $(H_\alpha) \subseteq \mathcal{T}$ such that

$$\limsup \|A + (\text{Id} - H_\alpha)\| = 1. \quad (3.1)$$

To simplify subsequent calculations, let us write the standard basis of $Z$ as $\{e_{nl} : n \in \mathbb{N}, 1 \leq l \leq n\}$ and let $\{e^*_{nl} : n \in \mathbb{N}, 1 \leq l \leq n\}$ be the corresponding biorthogonal functionals. Then $Ae_{nl} = e_{ml}$, where $m = (n - 1)n/2 + l$.

Given $0 < \varepsilon < 1$,

$$\max \|A + (\text{Id} - H_\alpha)\| < 1 + \varepsilon \quad (3.2)$$

for infinitely many $\alpha$’s. For such an $\alpha$ and every $e_{nl}$,

$$\max \|Ae_{nl} - (\text{Id} - H_\alpha)e_{nl}\| < 1 + \varepsilon. \quad (3.3)$$
Put $\alpha_{kj} = e_j^* (\text{Id} - H_\alpha) e_n$. Then,

$$
    \max \| \pm A e_n \pm (\text{Id} - H_\alpha) e_n \|_p
    = \max \| \pm e_m - (\text{Id} - H_\alpha) e_n \|_p
    = \left( \max_{\pm} |\alpha_{m_1} \pm 1| + |\alpha_{m_2} \pm 1| + \cdots + |\alpha_{m_l} \pm 1| \right)^p + \sum_{k \neq m} \left( \sum_{j = 1}^k |\alpha_{kj}| \right)^p
    < (1 + \varepsilon)^p.
$$

(3.4)

Since $\max_{\pm} |\alpha_{m_1} \pm 1| \geq 1$, it follows that $\sum_{k \neq m} \left( \sum_{j = 1}^k |\alpha_{kj}| \right)^p < (1 + \varepsilon)^p - 1$ and $|\alpha_{m_2} \pm 1| + \cdots + |\alpha_{m_l} \pm 1| < \varepsilon$. Since $\sqrt{1 + |\alpha_{m_1} \pm 1|} \leq \max_{\pm} |\alpha_{m_1} \pm 1| < 1 + \varepsilon$, $|\alpha_{m_1} | < \sqrt{\varepsilon} + \varepsilon^2 < 2 \sqrt{\varepsilon}$. Thus $\| (\text{Id} - H_\alpha) e_n \| < (3 \sqrt{\varepsilon})^p + (1 + \varepsilon)^p - 1)^{1/p} - 0$ as $\varepsilon \to 0$ uniformly in $n$ and $l$. It follows that, for any $n$,

$$
    \| P_n (\text{Id} - H_\alpha) P_n \| \leq \| P_n (\text{Id} - H_\alpha) j_n \| \leq ((3 \sqrt{\varepsilon})^p + (1 + \varepsilon)^p - 1)^{1/p},
$$

(3.5)

where $P_n$ is the projection on $Z$ with range $\ell_1^n \subseteq Z$ and $j_n$ is the canonical injection of $\ell_1^n$ into $Z$.

By Lemma 3.2 in [10], there exists a sequence $(k_n)$ such that, for the canonical projection $P$ from $Z$ onto $(\sum_{n = 1}^\infty \ell_1^{k_n})_p$,

$$
    \| P - PH_\alpha P \| = \| P (\text{Id} - H_\alpha) P \| < 3 ((3 \sqrt{\varepsilon})^p + (1 + \varepsilon)^p - 1)^{1/p}.
$$

(3.6)

Since $PH_\alpha P \in \mathcal{F}$ and $\varepsilon > 0$ is arbitrary small, by Lemmas 2.2 and 2.3, $\mathcal{F} = L(Z)$.

From Lemma 3.1 and Theorem 3.3, we have the following.

**COROLLARY 3.4.** If $\mathcal{F}$ is a proper ideal and also an $M$-ideal in $L(Z)$, then $\mathcal{F} = K(Z)$.

**Remark.** By duality, all the lemmas, Theorem 3.3 and Corollary 3.4 hold with $Z^\ast = (\sum_{n = 1}^\infty \ell_{1,\infty}^n)_p$, $1 < p < \infty$, in place of $Z$.

**Acknowledgement.** This research is supported by KOSEF No. 951-0102-003-2.

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