BANACH-MACKEY, LOCALLY COMPLETE SPACES, AND $\ell_{p,q}$-SUMMABILITY

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ABSTRACT. We defined the $\ell_{p,q}$-summability property and study the relations between the $\ell_{p,q}$-summability property, the Banach-Mackey spaces and the locally complete spaces.

We prove that, for $c_0$-quasibarrelled spaces, Banach-Mackey and locally complete are equivalent. Last section is devoted to the study of CS-closed sets introduced by Jameson and Kakol.

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1. Introduction. Let $(E, \tau)$ be a locally convex space. If $A$ is absolutely convex its linear span $E_A$ may be endowed with the seminorm topology given by the Minkowski functional of $A$, we denote it by $(E_A, \rho_A)$. If $A$ is bounded then $(E_A, \rho_A)$ is a normed space. If every bounded set $B$ is contained in an absolutely convex, closed, bounded set, called a disk $A$ such that $(E_A, \rho_A)$ is complete (barrelled) then $E$ is said to be locally complete (barrelled).

A locally convex space is a Banach-Mackey space if $\sigma(E, E')$-bounded sets are $\beta(E, E')$-bounded sets.

Finally, let us define the $\ell_{p,q}$-summability property. For $1 \leq p \leq \infty$ let $q$ be such that $(1/p) + (1/q) = 1$. A sequence $(x_n)_n \subset E$ is $p$-absolutely summable if for every $\rho$ continuous seminorm in $(E, \tau)$ the sequence $(\rho(x_n))_n$ is in $\ell_p$. A $p$-absolutely summable sequence is $\ell_{p,q}$-summable if for every $(\lambda_n)_n \in \ell_q$, the series $\sum_{n=1}^{\infty} \lambda_n x_n$ converges to $x$ for some $x \in E$. A locally convex space $E$ has the $\ell_{p,q}$-summability property if each $p$-absolutely summable sequence is $\ell_{p,q}$-summable.

2. $\ell_{p,q}$-summability. Let $(E, \tau) = (c_0, \sigma(c_0, \ell_1))$. $(E, \tau)$ is a locally complete space. Take $\alpha = (\alpha_n)_n \in \ell_1$ and $(e_n)_n$ the canonical unit vectors in $c_0$. Then $\rho_\alpha(e_n) = |\alpha_n|$ so $\sum_{n=1}^{\infty} \rho_\alpha(e_n) = \sum_{n=1}^{\infty} |\alpha_n| < \infty$ which means that $(e_n)_n$ is absolutely summable for every continuous seminorm in $\sigma(c_0, \ell_1)$. Now, since $\sum_{n=1}^{\infty} (e_n) \notin c_0$ we have here an example of a space that has the $\ell_{\infty,1}$-summability property and does not have the $\ell_{1,\infty}$-summability property.

Now let us establish some properties of the spaces with the $\ell_{p,q}$-summability property.
**Theorem 2.1.** Let \((E, \tau)\) be a locally convex space. If \(E\) satisfies the \(\ell_{p,q}\)-summability property for \(1 \leq p, q \leq \infty\) with \((1/p) + (1/q) = 1\), then \(E\) is locally complete.

**Proof.** Let \(A\) be a bounded set and \(B = \overline{\text{conv}} A; B\) is a disk. Take \((x_n)_n \in E\) a sequence such that \((\rho_B(x_n))_n \in \ell_p\). Since \(i: (E_B, \rho_B) \hookrightarrow (E, \tau)\) is continuous, for every continuous seminorm \(\rho\) in \(E\), we have \((\rho(x_n))_n \in \ell_p\). So for every \((a_n)_n \in \ell_q\), we have \(\sum_{n=1}^{\infty} a_n x_n \to x\) with respect to \(\tau\) since \(E\) has the \(\ell_{p,q}\)-summability property.

Now the sequence of partial sums \(\sum_{n=1}^{k} a_n x_n\) is \(\rho_B\)-bounded since it is a \(\rho_B\)-Cauchy sequence as we can see

\[
\rho_B \left( \sum_{n=1}^{k+r} a_n x_n - \sum_{n=1}^{k} a_n x_n \right) = \rho_B \left( \sum_{k+1}^{k+r} a_n x_n \right) \leq \left| (a'_n)_n \right|_q \cdot \left| (\rho_B(x'_n))_n \right|_p, \quad (2.1)
\]

which is small for \(k\) big enough, \((a'_n)_n = (0, \ldots, 0, a_{k+1}, \ldots, a_{k+r}, 0, \ldots)\) and \((x'_n)_n = (0, \ldots, 0, x_{k+1}, \ldots, x_{k+r}, 0, \ldots)\).

So \(\{\sum_{n=1}^{k} a_n x_n : K \in \mathbb{N}\}\) is a \(\rho_B\) bounded set in \((E_B, \rho_B)\).

By [5, Theorem 3.2.4] we have that \((\sum_{n=1}^{k} a_n x_n)_K\) converges to \(x\) in \((E_B, \rho_B)\). So \((E_B, \rho_B)\) has also the \(\ell_{p,q}\)-summability property.

Now, we will prove the space \((E_B, \rho_B)\) is complete. Let \((x_n)_n \in E\) be an absolutely summable sequence with \(x_n \neq 0\) for every \(n \in \mathbb{N}\), so \((\rho_B(x_n))_n \in \ell_1\) then

\[
(a_n)_n = \left( \rho_B^{1/p}(x_n) \right)_n \in \ell_p, \quad (\beta_n)_n = \left( \rho_B^{1/q}(x_n) \right)_n \in \ell_q. \quad (2.2)
\]

Let \(y_n = x_n / \rho_B(x_n)\) then \((y_n)_n\) is \(\rho_B\)-bounded. So \((a_n y_n)_n \in E_B, (\rho_B(\alpha_n y_n)_n) \in \ell_p\) and \(\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \alpha_n y_n\) converges in \((E_B, \rho_B)\) since \((E_B, \rho_B)\) has the \(\ell_{p,q}\)-summability property so \((E_B, \rho_B)\) is a Banach disk. \(\square\)

**Corollary 2.2.** Let \((E, \tau)\) be a locally convex space. \((E, \tau)\) is locally complete if and only if \((E, \tau)\) has the \(\ell_{\infty,1}\)-summability property.

**Proof.** Let \((E, \tau)\) be a locally complete space and \((x_n)_n \subset (E, \tau)\) be a bounded sequence, so there exists a Banach disk \(B \subset E\) such that \(\{x_n\}_n \subset B\) and \((x_n)_n\) is bounded in \((E_B, \rho_B)\).

Let \((\alpha_n)_n \in \ell_1,\) then \((\alpha_n x_n)_n\) is \(\rho_B\)-absolutely summable, that is \(\sum_{n=1}^{\infty} \alpha_n x_n < \infty\).

Hence \(\sum_{n=1}^{\infty} \alpha_n x_n\) converges in \((E_B, \rho_B)\) so it also converges in \((E, \tau)\) since \(i: (E_B, \rho_B) \hookrightarrow (E, \tau)\) is continuous. So \(E\) has the \(\ell_{\infty,1}\)-summability property. \(\square\)

**Corollary 2.3.** \(E\) is a Banach space if and only if \(E\) is normed and has the \(\ell_{p,q}\)-summability property.

**Proof.** We can reproduce the last part of the proof of Theorem 2.1 to show that \(E\) normed and with the \(\ell_{p,q}\)-summability property is a locally complete normed space and so a Banach space.

Now suppose \(E\) is a Banach space and denote the norm by \(\|\|\). Let \((x_n)_n \subset E\) be a sequence such that \((\|x_n\|)_n \in \ell_p\) and let \((\beta_n)_n \in \ell_1\) then the sequence \((\beta_n x_n)_n\) is absolutely summable that is
\[ \sum_{n=1}^{\infty} \| \beta_n x_n \| \leq \left( \sum_{n=1}^{\infty} \| x_n \|^p \right)^{1/p} \left( \sum_{n=1}^{\infty} \| \beta_n \|^q \right)^{1/q} < \infty \]  \hspace{1cm} (2.3)

hence summable, since \( E \) is a Banach space so \( E \) has the \( \ell_{p,q} \)-summability property. \( \square \)

3. Banach-Mackey space

**Definition 3.1.** \( E \) is a \( c_0 \)-barrelled (or \( c_0 \)-quasibarrelled) space if each null sequence in \((E', \sigma(E',E)) ((E', \beta(E',E)))\) is \( E \)-equicontinuous.

Note that a \( c_0 \)-barrelled space is a \( c_0 \)-quasibarrelled space.

**Lemma 3.2.** If \((E, \mu(E,E'))\) is a Banach-Mackey space, where \( \mu(E,E') \) denotes the Mackey topology, and \( c_0 \)-quasibarrelled space then it is a \( c_0 \)-barrelled space.

**Proof.** Let \( A \subset (E', \sigma(E',E)) \) be a bounded set, since \( E \) is a Banach-Mackey space, \( E' \) is also a Banach-Mackey space (cf. [9, Theorem 5, page 158]), and then \( A \) is \( \beta(E',E) \)-bounded so it is contained in a bounded Banach disk by [2, Observation 8.2.23], since the space is \( c_0 \)-quasibarrelled. Then by the same observation we have that \((E', \sigma(E',E))\) is locally complete. \( \square \)

**Corollary 3.3.** \((E, \mu(E,E'))\) is \( c_0 \)-quasibarrelled and Banach-Mackey if and only if \((E', \sigma(E',E))\) is locally complete.

**Proof.** Necessity follows from previous lemma and [2, Observation 8.2.23]. The other implication follows from the same observation, the note following Definition 3.1 and the fact that by [7, Corollary 3, Theorem 1] we have that \((E', \sigma(E',E))\) locally complete implies \((E, \mu(E,E'))\) is a Banach-Mackey space. \( \square \)

Following Saxon and Sánchez [8], a space \( E \) is dual locally complete if \((E, \sigma(E',E))\) is locally complete; then we can extend the result shown in [8, Theorem 2.6].

**Corollary 3.4.** \((E, \mu(E,E'))\) is dual locally complete if and only if it is Banach-Mackey and \( c_0 \)-quasibarrelled.

A locally convex space \( E \) is quasibarrelled if each barrel that absorbs bounded sets is a neighborhood of zero in \( E \). It is clear that a barrelled space is quasibarrelled, in certain cases they are equivalent.

Note that using [7, Theorem 1] we can easily prove that: a locally convex space \( E \) is quasibarrelled and Banach-Mackey if and only if it is a barrelled space. Next proposition summarizes what we know about Banach-Mackey spaces in the case of quasibarrelled spaces.

**Proposition 3.5.** Let \((E, \tau)\) be a locally convex quasibarrelled space, then the following properties are equivalent:

(a) \( E' \) is a Banach-Mackey space.

(b) \( E \) is a Banach-Mackey space.

(c) \( E \) is barrelled.

(d) \( E' \) is semireflexive.

(e) In \( E' \), abconv \( K \) is compact for each \( K \subset E' \) compact.
(f) For every $x_n \to 0$ in $E'$ and every $(\alpha_n)_n \in \ell_1$, $\sum_{n=1}^{\infty} \alpha_n x_n \to x$ for some $x \in E'$.

(g) $E'$ is locally complete.

(h) $E'$ is locally barrelled.

**Proof.** (a) $\Rightarrow$ (b) using [9, Theorem 5, page 158]. (b) $\Rightarrow$ (c) from the previous note. (c) $\Rightarrow$ (d) by [9, Theorem 4, page 153]. (d) $\Rightarrow$ (e) is obtained using the same theorem and the fact that a convex hull of a compact set is totally bounded together with [9, Exercise 5, page 122]. (e) $\Rightarrow$ (f) by [7, Theorems 2 and 3]. (f) $\Rightarrow$ (g) using [3, Proposition III.1.4] and [2, Theorem 5.1.11]. (g) $\Rightarrow$ (h) is trivial. (h) $\Rightarrow$ (a) using [1, Theorem 1].

Note that (f) and (g) are equivalent in general, [3, Proposition III.1.4] and [2, Theorem 5.1.11] prove (f) $\Rightarrow$ (g) and do not assume $E$ is quasibarrelled, and the other implication can be obtained using an argument similar to the one in Corollary 2.2.

4. CS-closed sets. In this section, we give a more precise definition of the convex series and their properties, first studied by Jameson [4] and Käkol [6].

**Definition 4.1.** Let $(E, \tau)$ be a locally convex space.

(a) Let $A \subset E$, $(a_n)_n \subset A$ and $(c_n) \subset [0,1]$ such that $\sum_{n=1}^{\infty} c_n = 1$ if $\sum_{n=1}^{\infty} c_n a_n$ is convergent we say that it is a convex convergent series of elements of $A$.

(b) $A \subset E$ is CS-closed if each convex convergent series of elements of $A$ belongs to $A$.

(c) $A \subset E$ is CS-compact if each convex series of elements of $A$ converges to an element of $A$.

(d) $A \subset E$ is ultrabounded if each convex series of elements of $A$ is convergent in $E$.

(e) The CS-closure of $A$ is the intersection of all CS-closed sets that contain $A$.

**Observation.** (i) An ultrabounded set is bounded.

(ii) The intersection of CS-closed sets is a CS-closed set.

For convenience let us introduce another definition.

**Definition 4.2.** (a) $B \subset E$ is called a CS-barrel if it is absolutely convex, absorbent and CS-closed.

(b) $E$ is a locally CS-barrelled (barrelled) space if for each bounded set $A \subset E$ there exists a disk $B$ such that $A \subset B$ and $E_B$ is a CS-barrelled (barrelled) space, that is that each CS-barrel (barrel) is a neighborhood of zero.

Now several properties of barrels also hold for CS-barrels although the last sets are somehow “smaller” than the first sets.

It is clear that if $E$ is a CS-barrelled space then it is a barrelled space.

Now if $(E, \tau)$ is locally barrelled, then for each bounded set $A \subset E$ there exists a closed bounded disk $B$ such that $A \subset B \subset E$ and $(E_B, \rho_B)$ is barrelled, so for each CS-barrel $U$ in $E_B$, $\overline{U}$ is a barrel so it is a zero neighborhood with respect to $\rho_B$, since $(E_B, \rho_B)$ is metrizable by [4, Theorem 1], $U$ is also a zero neighborhood with respect to $\rho_B$. So we have proved the following.

**Proposition 4.3.** $(E, \tau)$ is a locally barrelled space if and only if it is locally CS-barrelled space.
The CS-compact hull of a set $A$ is the set of convex convergent series of its elements. $A$ is CS-compact if each convex series of elements of $A$ converges to an element of $A$, so we have that the CS-compact hull of a set is not necessarily a CS-compact set. This is the moment to bring in the ultrabounded sets, since the CS-compact hull of an ultrabounded set is a CS-compact set.

**Proposition 4.4.** In a locally convex space $(E,\tau)$, CS-barrels absorb ultrabounded sets.

**Proof.** Let $W$ be a CS-barrel and $A$ an ultrabounded set in $E$. Let $D$ be the balanced CS-compact hull of $A$, by [6, Corollaries 2–4] $D$ is a Banach disk so $D$ is barrelled, and the identity map $i : E_D \to E$ is continuous so $W^\tau \cap E_D$ is a barrel in $(E_D,\rho_D)$, furthermore it is a neighborhood of zero in $D_D$, so $A \subset D \subset \lambda W \cap E_D$ for some $\lambda > 0$. Now for $(x_n)_n \subset W \cap E_D$ and $(a_n)_n \in [0,1]$, with $\sum_n a_n = 1$ such that $\sum_n a_n x_n \to x$ in $(E_D,\rho_D)$, since $W$ is a CS-barrel in $(E,\tau)$, we have $\sum_n a_n x_n \to x$ in $(E,\tau)$ and $x \in W$, then $x \in W \cap E_D$ and it is a CS-barrel in $(E_D,\rho_D)$. By [4, Theorem 1], $W \cap E_D$ and $W^\tau \cap E_D$ have the same interior with respect to $\rho_B$, so $A \subset D \subset \lambda (W \cap E_D) \subset \lambda W$. \hfill $\square$

**Remark 4.5.** Since every Banach disk is ultrabounded (cf. [6, Proposition 2.2]) then each CS-barrel absorbs Banach disks.

To close this section let us mention that if $E$ is locally barrelled then each CS-barrel is a bornivorous (see [7, proof of Theorem 2(1)]).

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**References**


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