ANALOGUES OF SOME FUNDAMENTAL THEOREMS OF SUMMABILITY THEORY

RICHARD F. PATTERSON

(Received 18 February 1998)

ABSTRACT. In 1911, Steinhaus presented the following theorem: if \( A \) is a regular matrix then there exists a sequence of 0’s and 1’s which is not \( A \)-summable. In 1943, R. C. Buck characterized convergent sequences as follows: a sequence \( x \) is convergent if and only if there exists a regular matrix \( A \) which sums every subsequence of \( x \). In this paper, definitions for "subsequences of a double sequence" and "Pringsheim limit points" of a double sequence are introduced. In addition, multidimensional analogues of Steinhaus’ and Buck’s theorems are proved.

Keywords and phrases. Subsequences of a double sequence, Pringsheim limit point, P-convergent, P-divergent, RH-regular.

2000 Mathematics Subject Classification. Primary 40B05.

1. Introduction. In [2, 3, 4, 5, 8], the 4-dimensional matrix transformation \((Ax)_{m,n} = \sum_{k,l=0}^{\infty} a_{m,n,k,l}x_{k,l}\) is studied extensively by Robison and Hamilton. Here we define new double sequence spaces and consider the behavior of 4-dimensional matrix transformations on our new spaces. Such a 4-dimensional matrix \( A \) is said to be RH-regular if it maps every bounded P-convergent sequence (defined below) into a P-convergent sequence with the same P-limit. In [9] Steinhaus proved the following theorem: if \( A \) is a regular matrix then there exists a sequence of 0’s and 1’s which is not \( A \)-summable. This implies that \( A \) cannot sum every bounded sequence. In this paper, we prove a theorem for double sequences and 4-dimensional RH-regular matrices that is analogous to Steinhaus’ theorem. One of the fundamental facts of sequence analysis is that if a sequence is convergent to \( L \), then all of its subsequences are convergent to \( L \). In a similar manner, R. C. Buck [1] characterized convergent sequences by: a sequence \( x \) is convergent if and only if there exists a regular matrix \( A \) which sums every subsequence of \( x \). We characterize P-convergent double sequences as follows: first, we prove that a double sequence \( x \) is P-convergent to \( L \) if all of its subsequences are P-convergent to \( L \); then we prove that a double sequence \( x \) is P-convergent if there exists an RH-regular matrix \( A \) which sums every subsequence of \( x \). In addition, we provide definitions for “subsequences” and “Pringsheim limit points” of double sequences and for divergent double sequence.

2. Definitions, notations, and preliminary results

DEFINITION 2.1 (Pringsheim, 1900). A double sequence \( x = [x_{k,l}] \) has Pringsheim limit \( L \) (denoted by \( \lim_{n} x = L \)) provided that given \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that
\[ |x_{k,l} - L| < \epsilon \text{ whenever } k,l > N. \] We describe such an \( x \) more briefly as “P-convergent.”

**Definition 2.2** (Pringsheim, 1900). A double sequence \( x \) is called definite divergent, if for every (arbitrarily large) \( G > 0 \) there exist two natural numbers \( n_1 \) and \( n_2 \) such that \( |x_{n,k}| > G \) for \( n \geq n_1, \ k \geq n_2 \).

**Definition 2.3.** The sequence \( y \) is a subsequence of the double sequence \( x \) provided that there exist two increasing double index sequences \( \{ n_i \} \) and \( \{ k_i \} \) such that

\[
n_i^0 = k_i^0 = n_i^{0-1} = k_i^{0-1} = 0 \text{ and } \\
n_i^1 \text{ and } k_i^1 \text{ are both chosen such that } \max\{n_i^{2i-3}, k_i^{2i-3}\} < n_i^1, k_i^1, \\
n_i^2 \text{ and } k_i^2 \text{ are both chosen such that } \max\{n_i^1, k_i^1\} < n_i^2, k_i^2, \\
n_i^3 \text{ and } k_i^3 \text{ are both chosen such that } \max\{n_i^2, k_i^2\} < n_i^3, k_i^3, \\
\vdots \\
n_i^{2(i-1)} \text{ and } k_i^{2(i-1)} \text{ are both chosen such that } \max\{n_i^i, k_i^i\} < n_i^{2(i-1)}, k_i^{2(i-1)}, \text{ with}
\]

\[
\begin{align*}
Y_{1,i} &= x_{n_i^1, k_i^1}, & Y_{2,i} &= x_{n_i^2, k_i^2}, & Y_{3,i} &= x_{n_i^3, k_i^3}, \\
&\vdots & & & \\
Y_{i-1,i} &= x_{n_i^{i-1}, k_i^{i-1}}, & Y_{i,i} &= x_{n_i^i, k_i^i}, \\
&\vdots & & & \\
Y_{i,2i-1} &= x_{n_i^{2i-1}, k_i^{2i-1}}
\end{align*}
\]

for \( i = 1, 2, 3, \ldots \).

A double sequence \( x \) is bounded if and only if there exists a positive number \( M \) such that \( |x_{k,l}| < M \) for all \( k \) and \( l \). Define

\[
\begin{align*}
S'' \{ x \} &= \{ \text{all subsequences of } x \}; \\
C'' &= \{ \text{all bounded P-convergent sequences} \}; \\
C''_A &= \left\{ x_{k,l} : (Ax)_{m,n} = \sum_{k,l=0}^{\infty} a_{m,n,k,l} x_{k,l} \text{ is P-convergent} \right\}.
\end{align*}
\]

See Figure 1 for an illustration of the procedure for selecting terms of a subsequence. A 2-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The Silverman-Toeplitz theorem [6] characterizes the regularity of 2-dimensional matrix transformations. In 1926, Robison presented a 4-dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. The definition of the regularity for 4-dimensional matrices will be stated below, with the Robison-Hamilton characterization of the regularity of 4-dimensional matrices.
**Figure 1.** The selection process of terms for subsequence $y$ of $x$, where $x[n(i,j), k(i,j)] = x_{n^i_j, k^i_j}$, $n(i,j) = n^i_j$, $k(i,j) = k^i_j$.

**Definition 2.4.** The 4-dimensional matrix $A$ is said to be RH-regular if it maps every bounded $P$-convergent sequence into a $P$-convergent sequence with the same $P$-limit.

**Theorem 2.1** (Hamilton [2], Robison [8]). The 4-dimensional matrix $A$ is RH-regular if and only if

- RH$_1$: $P$-lim$_{m,n} a_{m,n,k,l} = 0$ for each $k$ and $l$;
- RH$_2$: $P$-lim$_{m,n} \sum_{k,l=0}^{\infty} a_{m,n,k,l} = 1$;
- RH$_3$: $P$-lim$_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$ for each $l$;
- RH$_4$: $P$-lim$_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$ for each $k$;
- RH$_5$: $\sum_{k,l=0}^{\infty} |a_{m,n,k,l}|$ is $P$-convergent;
- RH$_6$: there exist finite positive integers $A$ and $B$ such that $\sum_{k,l=B}^{\infty} |a_{m,n,k,l}| < A$.

**Remark 2.1.** The definition of a Pringsheim limit point can also be stated as follows: $\beta$ is a Pringsheim limit point of $x$ provided that there exist two increasing index sequences $\{n_i\}$ and $\{k_i\}$ such that $\lim_{i} x_{n_i, k_i} = \beta$.

**Definition 2.5.** A double sequence $x$ is divergent in the Pringsheim sense ($P$-divergent) provided that $x$ does not converge in the Pringsheim sense ($P$-convergent).
Remark 2.2. Definition 2.5 can also be stated as follows: a double sequence $x$ is $P$-divergent provided that either $x$ contains at least two subsequences with distinct finite limit points or $x$ contains an unbounded subsequence. Also note that, if $x$ contains an unbounded subsequence then $x$ also contains a definite divergent subsequence.

Remark 2.3. For an ordinary single-dimensional sequence, any sequence is a subsequence of itself. This, however, is not the case in the 2-dimensional plane, as illustrated by the following example.

Example 2.1. The sequence

$$x_{n,k} := \begin{cases} 1, & \text{if } n = k = 0, \\ 1, & \text{if } n = 0, k = 1, \\ 1, & \text{if } n = 1, k = 0, \\ 0, & \text{otherwise} \end{cases} \quad (2.2)$$

contains only two subsequences, namely, $[y_{n,k}] = 0$ for each $n$ and $k$, and

$$z_{n,k} := \begin{cases} 1, & \text{if } n = k = 0, \\ 0, & \text{otherwise}; \end{cases} \quad (2.3)$$

neither subsequence is $x$.

The following proposition is easily verified, and is worth stating since each single-dimensional sequence is a subsequence of itself. However, this is not the case for double-dimensional sequences.

Proposition 2.1. The double sequence $x$ is $P$-convergent to $L$ if and only if every subsequence of $x$ is $P$-convergent to $L$.

3. Main results. The next result is a “Steinhaus-type” theorem, so named because of its similarity to the Steinhaus theorem in [9] quoted in the introduction.

Theorem 3.1. If $A$ is an RH-regular matrix, then there exists a bounded double sequence $x$ consisting only of 0’s and 1’s which is not $A$-summable.

Proof. Let $m_i, n_j, k_i,$ and $l_j$ be increasing index sequences which we define as follows:

Let $k_0 := l_0 := -1$ and choose $m_0$ and $n_0$ such that $m_0, n_0 > B$, where $B$ is defined by RH$_6$ and RH$_2$ to imply

$$\left| \sum_{k,l=0}^{\infty} a_{m_0,n_0,k,l} \right| > \frac{1}{4}, \quad (3.1)$$

whenever $m_0, n_0 > B$.

Also, by RH$_1$, RH$_3$, RH$_4$, and RH$_5$ we choose $k_1 > k_0$ and $l_1 > l_0$ such that
ANALOGUES OF SOME FUNDAMENTAL THEOREMS

\[ \left| \sum_{k < k_1, l < l_1} a_{m_0, n_0, k, l} \right| > 1 - \frac{1}{4}, \]
\[ \sum_{k \geq k_1, l \geq l_1} |a_{m_0, n_0, k, l}| < \frac{1}{4}, \]
\[ \sum_{k \geq k_1, l < l_1} |a_{m_0, n_0, k, l}| < \frac{1}{4}, \]
\[ \sum_{k < k_1, l \geq l_1} |a_{m_0, n_0, k, l}| < \frac{1}{4}. \]  

(3.2)

Next use RH_1, RH_2, RH_3, and RH_4 to choose \( m_1 > m_0 \) and \( n_1 > n_0 \) such that
\[ \sum_{k < k_1, l < l_1} |a_{m_1, n_1, k, l}| < \frac{1}{9}, \]
\[ \sum_{k \geq k_1, l \geq l_1} |a_{m_1, n_1, k, l}| < \frac{1}{9}, \]
\[ \sum_{k \geq k_1, l \leq l_1} |a_{m_1, n_1, k, l}| < \frac{1}{9}, \]
\[ \left| \sum_{k, l} a_{m_1, n_1, k, l} \right| > 1 - \frac{1}{9}. \]

(3.3)

These inequalities imply
\[ \sum_{k > k_1, l > l_1} |a_{m_1, n_1, k, l}| > 1 - \frac{4}{9}, \]

(3.4)

because
\[ \left| \sum_{k > k_1, l > l_1} |a_{m_1, n_1, k, l}| \right| \geq - \sum_{k \leq k_1, l \leq l_1} |a_{m_1, n_1, k, l}| + 1 - \frac{1}{9} \]
\[- \sum_{k \geq k_1, l \geq l_1} |a_{m_1, n_1, k, l}| \]
\[- \sum_{k \leq k_1, l \leq l_1} |a_{m_1, n_1, k, l}|. \]

(3.5)

We now choose \( k_2 > k_1 \) and \( l_2 > l_1 \) such that
\[ \left| \sum_{k_1 < k < k_2, l_1 < l < l_2} a_{m_1, n_1, k, l} \right| > 1 - \frac{4}{9}, \]
\[ \sum_{k \geq k_2, l \geq l_2} |a_{m_1, n_1, k, l}| < \frac{1}{9}, \]
\[ \sum_{k_1 < k < k_2, l \geq l_2} |a_{m_1, n_1, k, l}| < \frac{1}{9}, \]
\[ \sum_{k \leq k_2, l_1 < l \leq l_2} |a_{m_1, n_1, k, l}| < \frac{1}{9}. \]  

(3.6)
In general, having
\begin{align*}
m_0 < \cdots < m_{i-1}, & \quad k_0 < \cdots < k_{i-1} < k_i, \\
n_0 < \cdots < n_{j-1}, & \quad l_0 < \cdots < l_{j-1} < l_j,
\end{align*}
(3.7)
we choose \( m_i > m_{i-1} \) and \( n_j > n_{j-1} \) such that by RH_1
\begin{equation}
\sum_{k \leq k_i, l \leq l_j} |a_{m_i,n_j,k,l}| < \frac{1}{(i+2)(j+2)},
\end{equation}
(3.8)
and by RH_3, RH_4
\begin{align*}
\sum_{k \leq k_i, l > l_j} |a_{m_i,n_j,k,l}| & < \frac{1}{(i+2)(j+2)}, \\
\sum_{k \geq k_i, l \leq l_j} |a_{m_i,n_j,k,l}| & < \frac{1}{(i+2)(j+2)}.
\end{align*}
(3.9)
In addition, by RH_2
\begin{equation}
\left| \sum_{k,l=0}^{\infty,\infty} a_{m_i,n_j,k,l} \right| > 1 - \frac{1}{(i+2)(j+2)},
\end{equation}
(3.10)
so
\begin{equation}
\sum_{k > k_i, l > l_j} |a_{m_i,n_j,k,l}| > 1 - \frac{4}{(i+2)(j+2)}.
\end{equation}
(3.11)
We now choose \( k_{i+1} > k_i \) and \( l_{j+1} > l_j \) such that
\begin{align*}
\left| \sum_{k_i < k < k_{i+1}, l_j < l < l_{j+1}} a_{m_i,n_j,k,l} \right| & > 1 - \frac{4}{(i+2)(j+2)}, \\
\sum_{k \geq k_{i+1}, l \geq l_{j+1}} |a_{m_i,n_j,k,l}| & < \frac{1}{(i+2)(j+2)}, \\
\sum_{k_i < k < k_{i+1}, l \geq l_{j+1}} |a_{m_i,n_j,k,l}| & < \frac{1}{(i+2)(j+2)}, \\
\sum_{k \geq k_{i+1}, l_j < l < l_{j+1}} |a_{m_i,n_j,k,l}| & < \frac{1}{(i+2)(j+2)}.
\end{align*}
(3.12)
Define \( x \) as follows:
\begin{equation}
x_{k,l} = \begin{cases} 
1, & \text{if } k_{2p} < k < k_{2p+1} \text{ and } l_{2t} < l < l_{2t+1} \text{ for } p, t = 0, 1, 2, \ldots, \\
0, & \text{otherwise}.
\end{cases}
\end{equation}
(3.13)
Let us label and partition \((AX)_{m_i,n_j}\) as follows:

\[
(AX)_{m_i,n_j} = \alpha_1 \sum_{0 \leq k \leq k_i, 0 \leq l \leq l_j} + \alpha_2 \sum_{0 \leq k \leq k_i, l_j \leq l} + \alpha_3 \sum_{k_{i+1} \leq k, l_{j+1} \leq l} + \alpha_4 \sum_{0 \leq l \leq l_j, k_{i+1} \leq k} + \alpha_5 \sum_{k_{i+1} \leq k, 0 \leq l \leq l_j} + \alpha_6 \sum_{0 \leq k \leq k_i \leq k_{i+1} \leq l_{j+1} \leq l} + \alpha_7 \sum_{k_{i+1} \leq k, l_j \leq l} + \alpha_8 \sum_{l_j \leq l \leq l_{j+1}, k_{i+1} \leq k} + \alpha_9 \sum_{k_{i+1} \leq k, l_j \leq l} + \alpha_{10} \sum_{k_{i+1} \leq k, l_{j+1} \leq l}
\]

\[
= \alpha_1 \sum_{0 \leq k \leq k_i, 0 \leq l \leq l_j} + \alpha_2 \sum_{0 \leq k \leq k_i, l_j \leq l} + \alpha_3 \sum_{k_{i+1} \leq k, l_{j+1} \leq l} + \alpha_4 \sum_{0 \leq l \leq l_j, k_{i+1} \leq k} + \alpha_5 \sum_{k_{i+1} \leq k, 0 \leq l \leq l_j} + \alpha_6 \sum_{0 \leq k \leq k_i \leq k_{i+1} \leq l_{j+1} \leq l} + \alpha_7 \sum_{k_{i+1} \leq k, l_j \leq l} + \alpha_8 \sum_{l_j \leq l \leq l_{j+1}, k_{i+1} \leq k} + \alpha_9 \sum_{k_{i+1} \leq k, l_j \leq l} + \alpha_{10} \sum_{k_{i+1} \leq k, l_{j+1} \leq l}
\]

where the general term \(a_{m_i,n_j,k,l}x_{k,l}\) is the same for each of the nine sums. Note that,

\[
|\alpha_4 + \alpha_5| \leq \frac{1}{(i+2)(j+2)},
\]

\[
|\alpha_2 + \alpha_6| \leq \frac{1}{(i+2)(j+2)}.
\]

**CASE 1.** If \(i\) and \(j\) are even, then

\[
\left| (AX)_{m_i,n_j} \right| > 1 - \frac{1}{(i+2)(j+2)} - |\alpha_1| - \cdots - |\alpha_8| > 1 - \frac{7}{(i+2)(j+2)},
\]

and the last expression has P-limit 1.

**CASE 2.** If at least one of \(i\) and \(j\) is odd, then \(\alpha_9 = 0\) and

\[
\left| (AX)_{m_i,n_j} \right| \leq |\alpha_1| + |\alpha_2| + \cdots + |\alpha_8| \leq \frac{6}{(i+2)(j+2)},
\]

and the last expression of (3.17) has P-limit 0. Thus the P-limit of \((AX)_{m,n}\) does not exist, and we have shown that an RH-regular matrix \(A\) cannot sum every double sequence, of 0’s and 1’s.

As with the original Steinhaus Theorem [9], we can state the following as an immediate consequence of Theorem 3.1.

**Corollary 3.1.** If \(A\) is an RH-regular matrix, then \(A\) cannot sum every bounded double sequence.

The next result is a “Buck-type” theorem.

**Theorem 3.2.** The bounded double complex sequence \(x\) is P-convergent if and only if there exists an RH-regular matrix \(A\) such that \(A\) sums every subsequence of \(x\).

**Proof.** Since every subsequence of a P-convergent sequence \(x\) is bounded and P-convergent, and \(A\) is an RH-regular matrix, then for such an \(x\) there exists an RH-regular matrix \(A\) such that \(S''\{x\} \subseteq C''\).

Conversely, we use an adaptation of Buck’s proof [1] to show that if \(A\) is any
RH-regular matrix and $x \notin C''$ then there exists a subsequence $y \in S'' \{x\}$ such that $Ay \notin C''$.

Note that every subsequence of $x$ is bounded and $x \notin C''$, which implies that $x$ has at least two distinct Pringsheim limit points, say $\alpha$ and $\beta$. Thus there exist increasing index sequences $\{n_j\}$ and $\{k_i\}$ such that $\limsup x_{n_j,k_i} = \alpha$ and $\liminf x_{n_j,k_i} = \beta$ with $\alpha \neq \beta$.

Now define

$$y = \frac{x - \beta}{\alpha - \beta}$$

which yields $\limsup y_{n_j,k_i} = 1$ and $\liminf y_{n_j,k_i} = 0$. As a result there exist two disjoint pairs of index sequences $\{\hat{n}_j, \hat{k}_j\}$ and $\{v_j, k_j\}$ such that the sequences $\hat{y}_1$ and $\hat{y}_2$ constructed using $\{\hat{n}_j, \hat{k}_j\}$ and $\{v_j, k_j\}$, respectively, have $P$-limits 1 and 0, respectively. Let

$$y_{n,k}^* := \begin{cases} 1, & \text{if } n = \hat{n}_j, k = \hat{k}_j, \\ 0, & \text{if } n = v_j, k = k_j, \\ y, & \text{otherwise.} \end{cases}$$

Hence, $\{y_{n,k}^*\}$ contains a subsequence $\{\hat{y}_i^*\}$ with infinitely many 0’s and 1’s, along its diagonal. This implies that $S'' \{\hat{y}^*\}$ contains all sequences of 0’s and 1’s. Thus by Theorem 3.1, there exists $\hat{y}^* \in S'' \{\hat{y}^*\}$ such that $A\hat{y}^* \notin C''$. Also, $P \lim (y - y^*)_{i,j} = 0$. We now select a subsequence $\{\tilde{y}_{i,j}\}$ of $\{y_{i,j}\}$ with terms satisfying $\limsup y_{n_i,k_i} = 1$ and $\liminf y_{n_i,k_i} = 0$ corresponding to the 0’s and 1’s, respectively of $\{\tilde{y}_{n,i,j}^*\}$. Therefore $P \lim (\tilde{y} - \tilde{y}^*)_{i,j} = 0$ and $\tilde{y}_{i,j} - \tilde{y}_{i,j}^*$ is bounded. By the linearity and regularity of $A, A(\tilde{y} - \tilde{y}^*)_{i,j} = (A\tilde{y})_{i,j} - (A\tilde{y}^*)_{i,j}$ and $P \lim A(\tilde{y} - \tilde{y}^*)_{i,j} = 0$. Now since $A\tilde{y}^* \notin C''$, it follows that $A\tilde{y} \notin C''$; and since $\tilde{y} = \tilde{x} - \beta/\alpha - \beta$, we have $A\tilde{x} \notin C''$.

**ACKNOWLEDGEMENT.** This paper is based on the author’s doctoral dissertation, written under the supervision of Prof. J. A. Fridy at Kent State University. I am extremely grateful to my advisor Prof. Fridy for his encouragement and advice.

**REFERENCES**


PATTERSON: DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, DUQUESNE UNIVERSITY, 440 COLLEGE HALL, PITTSBURGH, PA 15282, USA

E-mail address: pattersr@mathcs.duq.edu