SUBSEQUENCES AND CATEGORY

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ABSTRACT. If a sequence of functions diverges almost everywhere, then the set of subsequences which diverge almost everywhere is a residual set of subsequences.

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1. Introduction. In [1], Bilyeu, Lewis, and Kallman proved a general theorem about rearrangements of a series of Banach space valued functions. This theorem settled a question on rearrangements of Fourier series posed by Kac and Zygmund. Kallman [3] proved an analogous theorem for subseries of a series of Banach space valued functions. The purpose of this paper is to complete the cycle of these ideas by proving an analogous theorem (Theorem 1.1) for subsequences of a sequence of Banach space valued functions. Theorem 1.1 does not seem to follow directly from results of [1] or [3]. Other than [1, 3], the only precedent for Theorem 1.1 seems to be a paper [7] on subsequences of a sequence of complex numbers.

Let $S$ be the set of all sequences $s = (s_1, s_2, \ldots)$, where $1 \leq s_1 < s_2 < \cdots$ is a strictly increasing sequence of positive integers. $S$ is a closed subset of the countable product of the positive integers, and so $S$ is a complete separable metric space. Given any sequence of objects $a_1, a_2, \ldots$, one can identify the set of its subsequences both as a set and as a topological space with $S$. In this context, it is natural to identify a collection of subsequences with a subset of $S$ and ask if it is first category, second category, or residual ([5] or [6]). Define an equivalence relation $\sim$ on $S$ as follows: if $s, t \in S$, then $s \sim t$ if and only if $s_n = t_n$ for all sufficiently large $n$. Intuitively this states that $s \sim t$ if and only if $s$ and $t$ agree from some point on. It is simple to check that any nonempty subset of $S$ which is saturated with respect to $\sim$ is dense.

The main result of this paper is the following theorem, which is proved in Section 2.

**Theorem 1.1.** Let $(X, \mu)$ be a regular locally compact $\sigma$-finite measure space, $Z$ a separable Banach space, and $f_n : X \to Z$ a sequence of Borel measurable functions. Suppose that the sequence $f_n(x)$ diverges for $\mu$-a.e., $x \in X$. Then $\{s \in S \mid f_{s_n}(x) \text{ diverges for } \mu\text{-a.e. } x \in X\}$ is a residual set in $S$.

Just as in [1, 3], this measure-category result has a category-category analog which is discussed in Section 3.

2. Proof of Theorem 1.1. The following special case of Theorem 1.1 will be proved first.
Lemma 2.1. Let $K$ be a compact Hausdorff space, $Z$ a Banach space, and $f_n : K \to Z$ a sequence of continuous functions, and $\delta > 0$. Suppose that for every $x \in K$ and positive integer $N$, there exists a pair of integers $n = n(x,N)$ and $m = m(x,N)$ so that $N \leq n \leq m$ and $\|f_m(x) - f_n(x)\| > \delta$. Then $\{ s \in S \mid f_{s_n}(x) \text{ diverges for every } x \in K \}$ is a residual set in $S$.

Proof. If $m,n$ is a pair of integers such that $1 \leq n \leq m$ and $s \in S$, let $g_{s,m,n} : K \to [0, +\infty)$ be defined by $g_{s,m,n}(x) = \|f_{m}(x) - f_{s_n}(x)\|$. $g_{s,m,n}$ is continuous. Consider

$$A = \bigcap_{N \geq 1} \bigcup_{N \leq n \leq m, N \leq m_p} \left\{ s \in S \mid \cup_{1 \leq i \leq p} g_{s,m_i,n_1}^{-1}((\delta, +\infty)) = K \right\}. \quad (2.1)$$

Fix $1 \leq n \leq m$ and $s \in S$. Then $V = \{ t \in S \mid t_m = s_m \text{ and } t_n = s_n \}$ is an open neighborhood of $s$ in $S$. Hence, if $t \in V$, then $g_{t,m,n} = g_{s,m,n}$. This in turn implies that $A$ is a $G_\delta$ subset of $S$. Furthermore, $A$ is saturated with respect to the equivalence relation $\sim$ and therefore is a dense $G_\delta$ if it is nonempty.

$A$ is nonempty since $t = (1, 2, 3, \ldots)$ is in $A$. To see this, fix $N \geq 1$. For $N \leq n \leq m$, let $U(m,n) = g_{s,m,n}^{-1}((\delta, +\infty))$. Note that the collection $\{U(m,n)\}_{N \leq n \leq m}$ is an open covering of $K$ by hypothesis and so has a finite subcover, say $U(m_1, n_1), \ldots, U(m_p, n_p)$. One easily concludes from this that $t \in A$.

Finally, note that the Cauchy criterion for convergence implies that if $s \in A$, then $f_{s_n}(x)$ diverges for every $x \in K$. Hence, $A \subseteq \{ s \in S \mid f_{s_n}(x) \text{ diverges for every } x \in K \}$. This proves Lemma 2.1.

Proof of Theorem 1.1. We may assume that $\mu$ is a probability measure since $\mu$ is $\sigma$-finite. If $q \geq 1$, let

$$D_q = \bigcap_{N \geq 1} \bigcup_{N \leq n \leq m} \left\{ x \in X \mid \|f_m(x) - f_n(x)\| > \frac{1}{q} \right\}. \quad (2.2)$$

Each $D_q$ is a Borel subset of $X$, $D_q \subseteq D_{q+1}$, and the Cauchy criterion for convergence implies that $\cup_{q \geq 1} D_q = \{ x \in X \mid f_n(x) \text{ diverges} \}$. $\mu(\cup_{q \geq 1} D_q) = 1$ by assumption. Use a vector-valued version of Lusin’s Theorem [2] to choose, for each $q$, a compact subset $K_q$ of $D_q$ so that each $f_n | K_q$ is continuous and $\mu(D_q - K_q) < 1/q$. $R_q = \{ s \in S \mid f_{s_n}(x) \text{ diverges for every } x \in K_q \}$ is a residual subset of $S$ by Lemma 2.1. Hence, $R = \cap_{q \geq 1} R_q$ is a residual set in $S$ and is contained in $\{ s \in S \mid f_{s_n}(x) \text{ diverges for } \mu\text{-a.e., } x \in X \}$ since $\mu(\cup_{q \geq 1} K_q) = 1$. This proves Theorem 1.1.

3. Sequences of functions with the Baire property. Theorem 1.1 may be regarded as a measure-category result. The purpose of this section is to prove a category-category analog of Theorem 1.1 (cf. [1, Thm. 1.2] and [3, Thm. 3.1]).

Let $X$ be a Polish space. A subset of $X$ is said to have the Baire property if there exists an open set $U$ in $X$ so that $A \cap U$ is first category. The collection of all subsets of $X$ with the Baire property is a $\sigma$-algebra which includes the analytic sets in $X$. Let $Z$ be any other Polish space. A function $f : X \to Z$ is said to have the Baire property if $U$ open in $Z$ implies that $f^{-1}(U)$ has the Baire property in $X$. Any Borel function $f : X \to Z$ is a function with the Baire property. See [4, 5] or [6] for a thorough discussion of this circle of ideas. The following theorem is then a category-category analog of Theorem 1.1.
**Theorem 3.1.** Let $X$ be a Polish space, $Z$ a separable Banach space, and $f_n : X \to Z$ a sequence of functions with the Baire property. Suppose that $\{x \in X \mid f_n(x) \text{ diverges}\}$ is a residual subset of $X$. Then $\{s \in S \mid f_{s_n}(x) \text{ diverges on a residual subset of } X\}$ is a residual subset of $S$.

The following proposition, of independent interest, is needed to prove Theorem 3.1.

**Proposition 3.2.** Let $Z$ be a Banach space and let $\{z_n\}_{n \geq 1}$ be a sequence in $Z$. Let $A = \{s \in S \mid z_{s_n} \text{ converges}\}$. Then either $A = S$ or $A$ is of first category in $S$.

**Proof.** For $k \geq 1$ define

$$B_k = \bigcap_{N \geq 1} \bigcup_{N \leq n \leq m} \left\{ s \in S \mid \|z_{s_m} - z_{s_n}\| > \frac{1}{k} \right\}. \quad (3.1)$$

Note that $B_k \subseteq B_{k+1}$. Each set in square brackets is open in $S$. Hence, this formula shows that $B_k$ is a $G_\delta$. $B_k$ is dense if it is nonempty since it is saturated with respect to the equivalence relation $\sim$. Therefore, $B_k$ is a residual set in $S$ if it is nonempty since any dense $G_\delta$ is residual.

The Cauchy criterion for convergence implies that $A^c = \bigcup_{k \geq 1} B_k$. Hence, either $A = S$ or $A^c$ is residual in $S$; or either $A = S$ or $A$ is of first category in $S$. This proves Proposition 3.2.

**Proof of Theorem 3.1.** Check that the mapping $(x,s) \mapsto f_{s_n}(x)$, $X \times S \to Z$, is a function with the Baire property for every $n \geq 1$. Hence,

$$B = \{(x,s) \mid f_{s_n}(x) \text{ diverges}\} = \bigcup_{k \geq 1} \bigcap_{N \geq 1} \bigcup_{N \leq n \leq m} \left\{ (x,s) \mid \|f_{s_m}(x) - f_{s_n}(x)\| > \frac{1}{k} \right\} \quad (3.2)$$

is a subset of $X \times S$ with the Baire property. For each $x \in X$, let $B^c_x$ be the projection of $B^c \cap ((x) \times S)$ onto $S$. The hypotheses of Theorem 3.1 plus Proposition 3.2 imply that each $B^c_x$ is a first category subset of $S$, except for a first category set of $x$’s. But then $B^c$ is itself a first category subset of $X \times S$ [6, Thm. 15.4] and so $B^c$, the projection of $B^c \cap (X \times (s))$ onto $X$, is a first category subset of $X$, except for a first category set of $s$’s (Theorem of Kuratowski-Ulam, [6, Thm. 15.1]). Hence, $B_s$, the projection of $B \cap (X \times (s))$ onto $X$, is a residual subset of $X$ for all except a first category set of $s$’s. This proves Theorem 3.1.

**References**


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