REGULARITY OF CONSERVATIVE INDUCTIVE LIMITS

JAN KUCERA

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Abstract. A sequentially complete inductive limit of Fréchet spaces is regular, see [3]. With a minor modification, this property can be extended to inductive limits of arbitrary locally convex spaces under an additional assumption of conservativeness.

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Throughout the paper \( E_1 \subset E_2 \subset \cdots \) is a sequence of Hausdorff locally convex spaces with continuous identity maps \( id : E_n \to E_{n+1}, \ n \in \mathbb{N} \). Their respective topologies are denoted by \( \tau_n \). The topology of their inductive limit \( \text{ind} E_n \) is denoted by \( \tau = \text{ind} \tau_n \).

We will use a result from [1, Cor. IV. 6.5]. It reads:

If \( F \) as well as all spaces \( E_n \) are Fréchet and \( T : F \to \text{ind} E_n \) is a linear map with a closed graph, then there is \( n \in \mathbb{N} \) such that \( T \) is a continuous map of \( F \) into \( E_n \).

According to [2, Sec. 5.2], the space \( \text{ind} E_n \) is called \( \alpha \)-regular, resp. regular, if every set bounded in \( \text{ind} E_n \) is contained, resp. bounded, in some constituent space \( E_n \). We will need a slightly modified notion of regularity.

Definition 1. An inductive limit \( \text{ind} E_n \) is quasi \( \alpha \)-regular, resp. quasi regular, if every set bounded in \( \text{ind} E_n \) is a subset of a \( \tau \)-closure of a set contained, resp. bounded, in some constituent space \( E_n \).

Definition 2. An inductive limit \( \text{ind} E_n \) is called conservative if for every linear subspace \( F \subset \text{ind} E_n \), we have

\[
\text{ind} (F \cap E_n, \tau_n) = (F, \text{ind} \tau_n).
\]

Lemma. Let a locally convex (Hausdorff) space \( E \) be sequentially complete, and \( B \) be a balanced, bounded, closed, and convex set in \( E \). Then the linear span \( F \) of \( B \), equipped with the topology generated by the Minkowski functional of \( B \), is a Banach space and the identity map \( id : F \to E \) is continuous.

Proof. Clearly \( F \) is a normed space and \( id : F \to E \) is continuous.

To prove the completeness of \( F \), take a Cauchy sequence \( \{x_n\} \) in \( F \). Since \( id : F \to E \) is continuous, \( \{x_n\} \) is Cauchy in \( E \). Hence it converges to some \( x \in E \). The set \( \bigcup \{x_n ; n \in \mathbb{N}\} \), which is bounded in \( F \), is contained in some \( \alpha B \). Since the set \( \alpha B \) is closed in \( E \), we have \( x \in \alpha B \subset F \).

For any 0-nbhd \( \lambda B, \ \lambda > 0 \), in \( F \), there exists \( k \in \mathbb{N} \) such that \( m, n > k \) imply \( x_n - x_m \in \lambda B \). If we let \( m \to \infty \), we get \( x_n - x \in \lambda B \) for \( n > k \), i.e., \( x_n \to x \) in \( F \). \( \Box \)
Proposition 1. Any sequentially complete \( \text{ind} E_n \) is quasi \( \alpha \)-regular.

Proof. Let a set \( A \) be bounded in \( \text{ind} E_n \). Denote by \( B \) its balanced, convex, \( \tau \)-closed hull, and by \( F \) the linear span of \( B \) with the same topology \( \gamma \) as in the Lemma. We know that \( F \) is a Banach space.

For any \( n \in N \), denote by \( G_n \) the completion of the normed space \( (F \cap E_n, \gamma) \). Then \( G_n \subset F \) and \( F \) equals strict inductive limit \( \text{ind} G_n \). Since \( B \) is bounded in \( F \), it is bounded in \( \text{ind} G_n \). Hence, by [1, Cor. IV. 6.5], \( B \) is bounded in some \( G_n \).

Finally, \( A \subset B \) and \( B \) is a \( \gamma \)-closure of a set \( V = \bigcup \{E_n \cap \lambda B; 0 < \lambda < 1\} \) in \( F \cap E_n \). Hence \( A \) is also a subset of the \( \tau \)-closure of \( V \) in \( \text{ind} E_n \).

Proposition 2. Let \( \text{ind} E_n \) be sequentially complete and conservative. Then every set \( A \subset E_1 \), which is bounded in \( \text{ind} E_n \) is also bounded in some constituent space \( E_n \).

Proof. Take such \( A \) and assume that it is not bounded in any \( E_n \). Then for any \( n \in N \), there exists a balanced convex 0-nbhd \( U_n \) in \( E_n \) which does not absorb \( A \). For any \( m, n \in N \), choose \( a_{m,n} \in A \) such that \( a_{m,n} \notin mU_n \). Denote by \( B \) the \( \tau \)-closure of the convex balanced hull of \( \bigcup \{a_{m,n}; m, n \in N\} \) and by \( F \) the linear span of \( B \). For any \( m, n \in N \), there exists \( f_{m,n} \in (\text{ind} E_n)' \), (the dual of \( \text{ind} E_n \)), such that \( f_{m,n}(a_{m,n}) \neq 0 \). Put \( V_{m,n} = \{ x \in F; |f_{m,n}(x)| \leq 1 \} \) and denote by \( F_n \) the linear space \( F \) equipped with the topology generated by \( \{U_m; m \geq n\} \cup \{V_{m,n}; m, n \in N\} \). Then each \( F_n \) is a metrizable Hausdorff locally convex space and its completion \( G_n \) is a Fréchet space.

Finally, let \( H \) be the space \( F \) equipped with the topology generated by the Minkowski functional of \( B \). The set \( B \) is bounded in \( \text{ind} E_n \), hence, by the Lemma, \( H \) is Banach space and the identity map \( \text{id} : H \rightarrow \text{ind} E_n \) is continuous.

Since \( \text{ind} E_n \) is conservative and \( F \subset \text{ind} E_n \), we have

\[
\text{ind} (F, \tau_n) = (F, \text{ind} \tau_n). \tag{2}
\]

For any \( n \in N \), the identity maps \( (F, \tau_n) \rightarrow F_n \rightarrow G_n \) are continuous. Hence

\[
\text{id} : \text{ind} (F, \tau_n) \rightarrow \text{ind} G_n \tag{3}
\]

is continuous, too. Then, the continuity of \( \text{id} : H \rightarrow \text{ind} E_n \) implies the continuity of \( \text{id} : H \rightarrow (F, \text{ind} \tau_n) \). By (2) and (3), we finally get the continuity of \( \text{id} : H \rightarrow \text{ind} G_n \).

By [1, Cor. IV. 6.5], there exists \( n \in N \) such that \( \text{id} : H \rightarrow G_n \) is continuous. Since the set \( B \) is bounded in \( H \) and contained in \( F_n \), it is bounded in \( G_n \), and also bounded in \( F_n \). But then, \( B \), as well as its subset \( A \), are absorbed by the 0-nbhd \( V_n \) in \( F_n \), a contradiction.

By combining Propositions 1 and 2, we get

Theorem. Any sequentially complete conservative \( \text{ind} E_n \) is quasi regular.

Corollary. If moreover each space \( E_n \) in the above Theorem is closed in \( \text{ind} E_n \), then \( \text{ind} E_n \) is regular.

References


KUCERA: DEPARTMENT OF MATHEMATICS, WASHINGTON STATE UNIVERSITY, PULLMAN, WASHINGTON 99164-3113, USA