RESEARCH NOTES

A SIMPLE CHARACTERIZATION OF COMMUTATIVE $H^*$-ALGEBRAS

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Abstract. Commutative $H^*$-algebras are characterized without postulating the existence of Hilbert space structure.

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1. Introduction. Let $\mathfrak{M}$ be the space of all maximal regular ideals in a commutative $H^*$-algebra $A$ and let $x(M)$, $M \in \mathfrak{M}$, denote the Gelfand transform of $x$, Loomis [3] (in the sequel we use notation of Naimark [5]). Then it is easy to show (see Theorem 1 below) that the series $\sum x(M)\hat{y}(M)$ converges absolutely for all $x, y \in A$. Also, if we assume that each minimal self-adjoint idempotent in $A$ has norm one, then it is true that for each bounded linear function $f$ on $A$($f \in A^*$) there exists $a \in A$ such that $f(x) = \sum x(M)a(M)$ for all $x \in A$.

In this note we show that these properties could be used to characterize commutative proper $H^*$-algebras of this kind. More specifically we show that each semi-single completely symmetric, Naimark [5], Banach algebra with the above properties is a proper $H^*$-algebra with respect to some Hilbertian norm which is equivalent to its original norm. Also, there is a characterization of all proper commutative $H^*$-algebras.

2. Characterizations. Let $A$ be a complex commutative Banach algebra. We do not assume that $A$ has an identity and so, because of this, we have to deal with regular maximal ideals. An ideal $I$ in $A$ is regular if the algebra $A/I$ has an identity. If $M$ is maximal regular ideal then it is closed and the algebra $A/M$ is isomorphic to the complex field (Gelfand-Mazur theorem, complex case, Loomis [3, 22F]). It follows that there exists a continuous linear functional $F_M$, Loomis [3, 23B], such that $M = \{x \in A : F_M(x) = 0\}$, i.e., $M$ is the kernel (null space) of $F_M$.

The Gelfand transform $x()$ (we use the Naimark’s notion, Naimark [5], here) of $x$ is defined by setting $x(M) = F_M(x)$ (Loomis uses the notion $x^\wedge$ in Loomis [3, 23B]), where $M$ is a regular maximal ideal in $A$.

The algebra $A$ is said to be semi-simple if $\cap_{M \in \mathfrak{M}} M = \{0\}$ (as it is stated above, $\mathfrak{M}$ denotes the space of all maximal regular ideals as $A$). Equivalent condition: mapping $x \to x()$ is one to one. The algebra $A$ is said to be completely symmetric, Naimark [5],...
if it has an involution \( x \rightarrow x^* \) such that \( x^*(M) = \bar{x}(M) \) for all \( M \in \mathfrak{M} \).

More details of Gelfand theory could be found in Gelfand-Raikov-Silov [2], Loomis [3], Mackey [4], Naimark [5], Simmons [7], and others.

A proper \( H^* \)-algebra is a Banach algebra \( A \) with an involution \( x \rightarrow x^* \) and a scalar product \( (\, , ) \) such that \( (x, x) = \|x\|^2 \) and \( (xy, z) = (y, x^*z) = (x, y^*z) \) for all \( x, y, z \in A \). Note that \( A \) is semi-simple. For simplicity, a nonzero self-adjoint idempotent will be called projection (e.g., Saworotnow [6]). A projection \( e \) is minimal if it is not a sum of two projections whose product is zero.

A completely symmetric commutative Banach algebra is a Banach algebra with involution \( x \rightarrow x^* \) such that \( x^*(M) = \bar{x}(M) \) for all \( x \in A \) and \( M \in \mathfrak{M} \), Naimark [5, Sec. 14].

**Theorem 1.** Each proper commutative \( H^* \)-algebra \( A \) is completely symmetric in the sense of Naimark [5]. Also, the series \( \sum_{M \in \mathfrak{M}} |x(M)|^2 \) converges for each \( x \in A \) and if we assume that each minimal projection in \( A \) has norm one, then each bounded linear functional \( f \) on \( A(f \in \mathcal{A}^*) \) has the form \( f(x) = \sum x(M)\tilde{a}(M)(x \in A) \) for some \( a \in \mathcal{A} \).

**Proof.** First and second parts of the theorem follow from Loomis [3, 27G]. For each \( M \in \mathfrak{M} \) there exists a minimal projection \( e_M \) such that \( x(M) = (x, e_M)\|e_M\|^2 \), \( x = \sum_{M \in \mathfrak{M}} x(M) \times e_M \) and \( e_M e_M = 0 \) if \( M_1 \neq M_2 \) (Loomis [3] uses notation "\( e_a \)" instead of \( "e_M" \)). Note that \( \|e_M\| \geq 1 \) for each \( M \in \mathfrak{M} \) (\( \|e_M\| = \|e_M^*\| \leq \|e_M\|^2 \)).

It follows that \( \|x\|^2 = \sum_{M \in \mathfrak{M}} |x(M)|^2 \|e_M\|^2 \geq \sum_{M \in \mathfrak{M}} |x(M)|^2 \). The last part follows from Loomis [3, 10G]: If we assume that each minimal projection has norm one, then \( \|x\|^2 = \sum_{M \in \mathfrak{M}} |x(M)|^2 \) and \( (x, a) = \sum_{M \in \mathfrak{M}} x(M)\tilde{a}(M) \) for all \( x, a \in A \) (and there exists \( a \in A \) such that \( f(x) = (x, a) \) for all \( x \in A \)).

Now we have a characterization of those commutative \( H^* \)-algebra in which each minimal projection has norm one.

**Theorem 2.** Let \( A \) be a semi-simple commutative completely symmetric Banach algebra. Assume further that the series \( \sum_{M \in \mathfrak{M}} |x(M)|^2 \) converges for each \( x \in A \) and that for each bounded linear functional \( f \) on \( A \) there exists \( a \in A \) such that \( f(x) = \sum_{M \in \mathfrak{M}} x(M)\tilde{a}(M) \) for all \( x \in A \). Then there exists a Hilbertian norm \( \| \|_2 \) on \( A \), equivalent to the original norm such that \( A \) is an \( H^* \)-algebra with respect to the scalar product \( (\, , ) \) associated with \( \| \|_2 \) and the original involution. Also, each minimal projection in \( A \) has norm 1.

**Proof.** For each \( x, y \in A \), define \( (x, y) = \sum_{M \in \mathfrak{M}} x(M)\tilde{y}(M) \). This series converges absolutely for all \( x, y \in A \), since

\[
\sum_{i=1}^{k} |x(M_i)\tilde{y}(M_i)| \leq \frac{1}{2} \left( \sum_{i=1}^{k} |x(M_i)|^2 + \sum_{i=1}^{k} |y(M_i)|^2 \right)
\]

for each finite subset \( \{M_1, \ldots, M_k\} \) of \( \mathfrak{M} \). Hence, the inner product \( (\, , ) \) is defined everywhere on \( A \). Let \( \| \|_2 \) be the corresponding norm, \( \|x\|_2 = (x, x) \) for all \( x \in A \). Let us show that \( A \) is complete with respect to \( \| \|_2 \).

First, note that the completion \( A' \) of \( A \) with respect to \( \| \|_2 \) is a proper \( H^* \)-algebra (since \( \|x^*\|_2 = \|x\|_2 \) for all \( x \in A \)). Hence, \( A' \) is semi-simple. (It is a consequence of
Let us show that each \( x \in M \) there exists an \( a \in A \) if and this implies that \( Loomis [3, 27A] \). So we can apply [5, Sec. 12, Thm. 1]: there exists \( M > 0 \) such that \( \| a \|_2 \leq M \) for each \( n \). For each fixed \( x \in A \) define

\[
f(x) = \lim_{m \to \infty} \langle x, a_m \rangle.
\]

From \( |\langle x, a_m \rangle| < \| x \|_2 \| a_m \|_2 \leq M \| x \| \) we conclude that \( f \) is a bounded linear functional on \( A \). Hence, there exists \( a \in A \) so that \( f(x) = \sum_{M \in \mathbb{Z}} x(M) a(M) \) for each \( x \in A \).

Let us show that \( \| a - a_n \|_2 \to 0 \). Let \( \varepsilon > 0 \) be arbitrary, take \( n_0 \) so that \( \| a_m - a_n \|_2 < \varepsilon/2 \) if \( m, n > n_0 \). Let \( n > n_0 \) and \( x \in A \) be fixed. Then \( \| a - a_n \|_2^2 = |(a - a_n, a - a_n)| + |(a - a_n, a_m - a_n)| \leq |f(a - a_n) - (a - a_n, a_m)| + \| a - a_n \|_2 \| a_m - a_n \|_2 \).

Select \( m > n_0 \) so that

\[
|f(a - a_n) - (a - a_n, a_m)| \leq \frac{\varepsilon}{2} \| a - a_n \|_2.
\]

Thus

\[
\| a - a_n \|_2^2 \leq \frac{\varepsilon}{2} \| a - a_n \|_2 + \frac{\varepsilon}{2} \| a - a_n \|_2 = \varepsilon \| a - a_n \|_2,
\]

and this implies that \( \| a - a_n \|_2 < \varepsilon \) for each \( n > n_0 \). So, \( A \) is complete with respect to \( \| \cdot \|_2 \).

It follows from [5, Sec. 12, Thm. 1] that the norm \( \| \cdot \|_2 \) and the original norm \( \| \cdot \| \) on \( A \) are equivalent.

It is also easy to see that \( A \) is an \( H^* \)-algebra with respect to the scalar product \( (\ , \) \) (and the original involution).

Let us show that every minimal projection in \( A \) has norm one. First note that the product of any two distinct minimal projections \( e_1 \) and \( e_2 \) is zero, \( e_1 e_2 = 0 \). It follows from the fact that \( e = e_1 e_2 \) is also a projection and that \( e e_i = e_i \), \( i = 1, 2 \). This means that if \( e \neq 0 \), then both \( e = e_1 \) and \( e = e_2 \), which is impossible, since \( e_1 \neq e_2 \). Thus \( e_{M_1} e_{M_2} = 0 \) if \( M_1 \neq M_2 \) (as was remarked in a proof above). But this also means that every minimal projection \( e \) is of the form \( e = e_{M'} \) for some \( M' \in \mathbb{Z} \). It follows then that \( e(M') = 1 \) and \( e(M) = 0 \) if \( M \neq M' \). Thus \( \| e \|_2^2 = |e(M')|^2 = 1 \).

For the general case we have Theorems 3 and 4 below, which constitute a characterization of any proper commutative \( H^* \)-algebra. The characterization is stated in terms of multiplicative functionals (it could also be done in terms of ideals) (needless to say, Theorems 1 and 2 could be restated in terms of multiplicative functionals also).

**Theorem 3.** For each proper commutative \( H^* \)-algebra \( A \) there exists a real valued function \( k(q) \), defined on the set \( Q \) of all its continuous multiplicative linear functionals, with the following properties:

(i) \( k(q) \geq 1 \) for each \( q \in Q \),

(ii) The series \( \sum_{q \in Q} |q(x)|^2 k(q) \) converges for each \( x \in A \),

(iii) For each \( f \in A^* \), there exists \( \alpha \in A \) such that \( f(x) = \sum_{q \in Q} q(x) \alpha(q) k(q) \) for each \( x \in A \). \( (A^* \) denotes the dual of \( A \).\)
**Proof.** It is easy consequence of Loomis [3, 27G] that for each nonzero member \( q \) of \( Q \) there exists a unique minimal projection \( e_q \) such that \( q(x) = (x, e_q) \| e_q \|^2 \) and

\[
x = \sum_{q \in Q} q(x)e_q
\]  
(2.5)

for each \( x \in A \) (note that \( \{e_q\}_{q \neq 0} \) is an orthogonal basis for \( A \)). We define the function \( k(q) \) by setting \( k(q) = \| e_q \|^2 \) for each nonzero member \( q \) of \( Q \) and \( k(0) = 1 \). We leave it to the reader to verify that \( k(q) \) has desired properties. \( \square \)

**Theorem 4.** Let \( A \) be a semi-simple commutative completely symmetric algebra and let \( Q \) be the set of all its continuous multiplicative linear functionals. Assume that there exists a real valued function \( k(q) \) on \( Q \) with properties (i), (ii), and (iii) in Theorem 3.

Then \( A \) is an \( H^* \)-algebra with respect to some Hilbert space norm \( \| \|_2 \) equivalent to the original norm of \( A \), and the original involution.

**Proof.** Define the scalar product \( (\cdot, \cdot) \) on \( A \) by setting

\[
(x, y) = \sum_{q \in Q} q(x)q(y^*)k(q),
\]  
(2.6)

and take that corresponding norm \( \| \|_2 \) (with the property that \( (x, x) = \|x\|_2^2 \)). Then we proceed as in the proof of Theorem 2. \( \square \)

**References**


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