THE ABEL-TYPE TRANSFORMATIONS INTO \( \ell \)

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ABSTRACT. Let \( t \) be a sequence in \((0,1)\) that converges to 1, and define the Abel-type matrix \( A_{\alpha,t} \) by \( a_{nk} = \binom{k+\alpha}{k} t_n^{k+1} (1-t_n)^{\alpha+1} \) for \( \alpha > -1 \). The matrix \( A_{\alpha,t} \) determines a sequence-to-sequence variant of the Abel-type power series method of summability introduced by Borwein in [1]. The purpose of this paper is to study these matrices as mappings into \( \ell \). Necessary and sufficient conditions for \( A_{\alpha,t} \) to be \( \ell'-\ell, G-\ell, \) and \( Gw-\ell \) are established. Also, the strength of \( A_{\alpha,t} \) in the \( \ell-\ell \) setting is investigated.

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1. Introduction and background. The Abel-type power series method [1], denoted by \( A_{\alpha} \), \( \alpha > -1 \), is the following sequence-to-function transformation: if

\[
\sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k < \infty \quad \text{for } 0 < x < 1
\]

and

\[
\lim_{x \to 1^-} (1-x)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k = L,
\]

then we say that \( u \) is \( A_{\alpha} \)-summable to \( L \). In order to study this summability method as a mapping into \( \ell \), we must modify it into a sequence to sequence transformation. This is achieved by replacing the continuous parameter \( x \) with a sequence \( t \) such that \( 0 < t_n < 1 \) for all \( n \) and \( \lim t_n = 1 \). Thus, the sequence \( u \) is transformed into the sequence \( A_{\alpha,t}u \) whose \( n \)th term is given by

\[
(A_{\alpha,t}u)_n = (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k t_n^k.
\]

This transformation is determined by the matrix \( A_{\alpha,t} \) whose \( nk \)th entry is given by

\[
a_{nk} = \binom{k+\alpha}{k} t_n^k (1-t_n)^{\alpha+1}.
\]

The matrix \( A_{\alpha,t} \) is called the Abel-type matrix. The case \( \alpha = 0 \) is the Abel matrix introduced by Fridy in [5]. It is easy to see that the \( A_{\alpha,t} \) matrix is regular and, indeed, totally regular.
2. Basic notations. Let $A = (a_{nk})$ be an infinite matrix defining a sequence-to-sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,$$ \hspace{1cm} (2.1)

where $(Ax)_n$ denotes the $n$th term of the image sequence $Ax$. The sequence $Ax$ is called the $A$-transform of the sequence $x$. If $X$ and $Z$ are sets of complex number sequence, then the matrix $A$ is called an $X$-$Z$ matrix if the image $Au$ of $u$ under the transformation $A$ is in $Z$ whenever $u$ is in $X$.

Let $y$ be a complex number sequence. Throughout this paper, we use the following basic notations:

\[
\ell = \left\{ y : \sum_{k=0}^{\infty} |y_k| \text{ converges} \right\},
\]

\[
\ell^p = \left\{ y : \sum_{k=0}^{\infty} |y_k|^p \text{ converges} \right\},
\]

\[
d(A) = \left\{ y : \sum_{k=0}^{\infty} a_{nk} y_k \text{ converges for each } n \geq 0 \right\},
\]

\[
\ell(A) = \left\{ y : Ay \in \ell \right\},
\]

\[
G = \left\{ y : y_k = O(r^k) \text{ for some } r \in (0,1) \right\},
\]

\[
G_w = \left\{ y : y_k = O(r^k) \text{ for some } r \in (0,w), 0 < w < 1 \right\},
\]

\[
c(A) = \left\{ y : y \text{ is summable by } A \right\}.
\]

3. The main results. Our first result gives a necessary and sufficient condition for $A_{\alpha,t}$ to be $\ell$-$\ell$.

**Theorem 1.** Suppose that $-1 < \alpha \leq 0$. Then the matrix $A_{\alpha,t}$ is $\ell$-$\ell$ if and only if $(1-t)^{\alpha+1} \in \ell$.

**Proof.** Since $-1 < \alpha \leq 0$ and $0 < t_n < 1$, we have

$$\sum_{n=0}^{\infty} |a_{nk}| = \binom{k+\alpha}{k} \sum_{n=0}^{\infty} t_n^k (1-t_n)^{\alpha+1} \leq \sum_{n=0}^{\infty} (1-t_n)^{\alpha+1} \text{ for each } k.$$ \hspace{1cm} (3.1)

Thus, if $(1-t)^{\alpha+1} \in \ell$, Knopp-Lorentz theorem [6] guarantees that $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix. Also, if $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then by Knopp-Lorentz theorem, we have

$$\sum_{n=0}^{\infty} |a_{n,o}| < \infty,$$ \hspace{1cm} (3.2)

and this yields $(1-t)^{\alpha+1} \in \ell$. \hfill $\square$

**Remark 1.** In Theorem 1, the implication that $A_{\alpha,t}$ is $\ell$-$\ell$ $\Rightarrow$ $(1-t)^{\alpha+1} \in \ell$ is also true for any $\alpha > 0$, however, the converse implication is not true for any $\alpha > 0$. This is demonstrated in Theorem 4 below.
**Corollary 1.** If \(-1 < \alpha \leq 0\) and \(0 < t_n < w_n < 1\), then \(A_{\alpha,t}\) is an \(\ell - \ell\) matrix whenever \(A_{\alpha,t}\) is an \(\ell - \ell\) matrix.

**Proof.** The corollary follows easily by Theorem 1.

**Corollary 2.** If \(-1 < \alpha < \beta \leq 0\), then \(A_{\beta,t}\) is an \(\ell - \ell\) matrix whenever \(A_{\alpha,t}\) is an \(\ell - \ell\) matrix.

**Corollary 3.** If \(-1 < \alpha \leq 0\) and \(A_{\alpha,t}\) is an \(\ell - \ell\) matrix, then \(1 / \log(1 - t) \in \ell\).

**Corollary 4.** If \(-1 < \alpha \leq 0\), then \(\arcsin(1 - t)^{\alpha + 1} \in \ell\) if and only if \(A_{\alpha,t}\) is an \(\ell - \ell\) matrix.

**Corollary 5.** Suppose that \(-1 < \alpha \leq 0\) and \(w_n = 1/t_n\). Then the zeta matrix \(z_{w}\) [2] is \(\ell - \ell\) whenever \(A_{\alpha,t}\) is an \(\ell - \ell\) matrix.

**Corollary 6.** Suppose that \(-1 < \alpha \leq 0\) and \(t_n = 1 - (n + 2)^{-q}\), \(0 < q < 1\); then \(A_{\alpha,t}\) is not an \(\ell - \ell\) matrix.

**Proof.** Since \((1 - t)^{\alpha + 1}\) is not in \(\ell\), the corollary follows easily by Theorem 1.

Before considering our next theorem, we recall the following result which follows as a consequence of the familiar Hölder’s inequality for summation. The result states that if \(x\) and \(y\) are real number sequences such that \(x \in \ell^p, y \in \ell^q, p > 1\), and \((1/p) + (1/q) = 1\), then \(xy \in \ell\).

**Theorem 2.** If \(A_{\alpha,t}\) is an \(\ell - \ell\) matrix, then

\[
\sum_{n=0}^{\infty} \log \left( \frac{2 - t_n}{n + 1} \right) < \infty. \tag{3.3}
\]

**Proof.** Since \(\log(2 - t_n) \sim (1 - t_n)\), it suffices to show that

\[
\sum_{n=0}^{\infty} \frac{(1 - t_n)}{(n + 1)} < \infty. \tag{3.4}
\]

If \(A_{\alpha,t}\) is an \(\ell - \ell\) matrix, then, by Theorem 1, we have \((1 - t)^{\alpha + 1} \in \ell\). If \(-1 < \alpha \leq 0\), it is easy to see that if \((1 - t)^{\alpha + 1} \in \ell\), then we have \((1 - t) \in \ell\) and, consequently, the assertion follows. If \(\alpha > 0\), then the theorem follows using the preceding result by letting \(x_n = 1 - t_n, y_n = 1/(n + 1), p = \alpha + 1,\) and \(q = (\alpha + 1) / \alpha\).

**Theorem 3.** Suppose that \(t_n = (n + 1)/(n + 2)\). Then \(A_{\alpha,t}\) is an \(\ell - \ell\) matrix if and only if \(\alpha > 0\).

**Proof.** If \(A_{\alpha,t}\) is an \(\ell - \ell\) matrix, then, by Theorem 1, it follows that \((1 - t)^{\alpha + 1} \in \ell\) and this yields \(\alpha > 0\). Conversely, suppose that \(\alpha > 0\). Then we have

\[
\sum_{n=0}^{\infty} |a_{nk}| = \left( \frac{k + \alpha}{k} \right) \sum_{n=0}^{\infty} \frac{n + 1}{n + 2} (n + 2)^{-(\alpha + 1)} = \left( \frac{k + \alpha}{k} \right) \sum_{n=0}^{\infty} (n + 1)^k (n + 2)^{-(k + \alpha + 1)} \leq M \left( \frac{k + \alpha}{k} \right) \int_0^{\infty} (x + 1)^k (x + 2)^{-(k + \alpha + 1)} dx
\]

\[
(3.5)
\]
for some $M > 0$. This is possible as both the summation and the integral are finite since $\alpha > 0$. Now, we let

$$g(k) = \int_0^{\infty} (x+1)^k (x+2)^{-(k+\alpha)} \, dx,$$

(3.6)

and we compute $g(k)$ using integration by parts repeatedly. We have

$$g(k) = \frac{1}{k+\alpha} \cdot 2^{-(k+\alpha)} + h_1(k),$$

(3.7)

where

$$h_1(k) = \frac{k}{k+\alpha} \int_0^{\infty} (x+1)^{k-1} (x+2)^{-(k+\alpha)} \, dx$$

(3.8)

and

$$h_2(k) = \frac{k(k-1)}{(k+\alpha)(k+\alpha-1)} \int_0^{\infty} (x+1)^{k-2} (x+2)^{-(k+\alpha-1)} \, dx$$

(3.9)

It follows that

$$h_3(k) = \frac{k(k-1)(k-2) \cdot 2^{-(k+\alpha-3)}}{(k+\alpha)(k+\alpha-1)(k+\alpha-2)(k+\alpha-3)} + h_4(k),$$

(3.10)

where

$$h_4(k) = \frac{k(k-1)(k-2)(k-3)}{(k+\alpha)(k+\alpha-1)(k+\alpha-2)(k+\alpha-3)(k+\alpha-4)} \int_0^{\infty} (x+1)^{k-4} (x+2)^{-(k+\alpha-3)} \, dx.$$  

(3.11)

Continuing this process, we get

$$h_k(k) = \frac{k(k-1)(k-2) \cdots 2^{\alpha}}{(k+\alpha)(k+\alpha-1)(k+\alpha-2) \cdots \alpha} = \frac{2^{\alpha}}{\alpha\binom{k+\alpha}{k}}.$$  

(3.12)

It is easy to see that $g(k)$ can be written using summation notation as

$$g(k) = \frac{2^{\alpha}}{\alpha\binom{k+\alpha}{k}} \sum_{i=0}^{k} \binom{i+\alpha-1}{i} 2^{-i}$$

$$\leq \frac{2^{\alpha}}{\alpha\binom{k+\alpha}{k}} \sum_{i=0}^{\infty} \binom{i+\alpha-1}{i} 2^{-i}$$

(3.13)

$$= \frac{2^{\alpha}}{\alpha\binom{k+\alpha}{k}} 2^{\alpha} = \frac{1}{\alpha\binom{k+\alpha}{k}}.$$
Consequently, we get
\[
\sum_{n=0}^{\infty} |a_{nk}| \leq M \binom{k+\alpha}{k} g(k) \leq \frac{M \binom{k+\alpha}{k}}{\alpha^{k+\alpha}}.
\] (3.14)

Thus by the Knopp-Lorentz theorem [6], \(A_{\alpha,t}\) is an \(\ell-\ell\) matrix.

**Corollary 7.** Suppose \(t_n = \frac{(n+1)}{(n+2)}\). Then \(A_{\alpha,t}\) is an \(\ell-\ell\) matrix if and only if \((1-t)^{\alpha+1} \in \ell\).

**Theorem 4.** Suppose \(\alpha > 0\) and \(t_n = 1 - \frac{(n+2)^{-q}}{q}, 0 < q < 1\). Then \(A_{\alpha,t}\) is not an \(\ell-\ell\) matrix.

**Proof.** If \((1-t)^{\alpha+1}\) is not in \(\ell\), then by Theorem 1, \(A_{\alpha,t}\) is not \(\ell-\ell\). If \((1-t)^{\alpha+1} \in \ell\), then we prove that \(A_{\alpha,t}\) is not \(\ell-\ell\) by showing that the condition of the Knopp-Lorentz theorem [6] fails to hold. For convenience, we let \(q = 1/p\) and \(2^{1/p} = R\), where \(p > 1\). Then we have
\[
\sum_{n=0}^{\infty} |a_{nk}| = \left(\frac{k+\alpha}{k}\right) \sum_{n=0}^{\infty} \left(1 - \frac{(n+2)^{-1/p}}{q}\right)^k (n+2)^{-1/p} (n+2)^{(k+\alpha+1)}
\]
\[
= \left(\frac{k+\alpha}{k}\right) \sum_{n=0}^{\infty} \left(1 - \frac{(n+2)^{1/p}}{q}\right)^k (n+2)^{(k+\alpha+1)}
\] (3.15)
\[
\geq M \left(\frac{k+\alpha}{k}\right) \int_{0}^{\infty} \left(1 - \frac{(x+2)^{1/p}}{q}\right)^k (x+2)^{(k+\alpha+1)} \, dx
\]

for some \(M > 0\). This is possible as both the summation and integral are finite since \((1-t)^{\alpha+1} \in \ell\). Now, let us define
\[
g(k) = \int_{0}^{\infty} \left(1 - \frac{(x+2)^{1/p}}{q}\right)^k (x+2)^{(k+\alpha+1)} \, dx.
\] (3.16)

Using integration by parts repeatedly, we can easily deduce that
\[
g(k) = \frac{p(R-1)^{k+\alpha+1-p}}{k+\alpha+1-p} + \frac{pk(R-1)^{k-1-p}(R)^{-(\alpha+1-p)}}{(k+\alpha+1-p)(k+\alpha-p)}
\]
\[
+ \cdots + \frac{pk(k-1)(k-2)\cdots(R)^{-(\alpha+1-p)}}{(k+\alpha+1-p)(k+\alpha-p)(k+\alpha-1-p)\cdots(\alpha+1-p)}.
\] (3.17)

This implies that
\[
g(k) > \frac{pk(k-1)(k-2)\cdots R^{-(\alpha+1-p)}}{(k+\alpha+1-p)(k+\alpha-p)(k+\alpha-1-p)\cdots(\alpha+1-p)}
\]
\[
= \frac{pR^{-(\alpha+1-p)}}{(\alpha+1-p)^k}.
\] (3.18)
Now, we have
\[
\sum_{n=0}^{\infty} |a_{nk}| \geq M_1 \left( \frac{k+\alpha}{k} \right) g(k) > \frac{pM_1 \left( \frac{k+\alpha}{k} \right) R^{-(\alpha+1-p)}}{(\alpha+1-p) \left( \frac{k+\alpha+1}{k} \right)} > \frac{M_2 k^\alpha}{k^{\alpha+1-p}} = M_2 k^{p-1}.
\]

Thus, it follows that
\[
\sup_k \left\{ \sum_{n=0}^{\infty} |a_{nk}| \right\} = \infty,
\]
and hence \( A_{\alpha,t} \) is not \( \ell^\infty \). \( \square \)

In case \( t_n = 1 - (n+2)^{-q} \), it is natural to ask whether \( A_{\alpha,t} \) is an \( \ell^\infty \) matrix. For \(-1 < \alpha \leq 0\), it is easy to see that \( A_{\alpha,t} \) is \( \ell^\infty \) if and only if \( \alpha > (1-q)/q \), by Theorem 1. For \( \alpha > 0 \), the answer to this question is given by the next theorem, which gives a necessary and sufficient condition for the matrix to be \( \ell^\infty \).

**Theorem 5.** Suppose that \( \alpha > 0 \) and \( t_n = 1 - (n+2)^{-q} \). Then \( A_{\alpha,t} \) is an \( \ell^\infty \) matrix if and only if \( q \geq 1 \).

**Proof.** Suppose that \( q \geq 1 \). Let \( q = 1/p, 2^{1/p} = R \) and \( (R-1)/R = S \), where \( 0 < p \leq 1 \). Then we have
\[
\sum_{n=0}^{\infty} |a_{nk}| = \left( \frac{k+\alpha}{k} \right) \sum_{n=0}^{\infty} \left( 1 - (n+2)^{-1/p} \right)^k (n+2)^{(-1/p)(\alpha+1)}
\]
\[
= \left( \frac{k+\alpha}{k} \right) \sum_{n=0}^{\infty} \left( (n+2)^{1/p} - 1 \right)^k (n+4)^{(-1/p)(k+\alpha+1)} \leq M \left( \frac{k+\alpha}{k} \right) \int_0^{\infty} \left( (x+2)^{1/p} - 1 \right)^k (x+2)^{(-1/p)(k+\alpha+1)} \, dx
\]
for some \( M > 0 \). This is possible as both the summation and the integral are finite since \( (1-t)^{\alpha+1} \in \ell \) for \( \alpha > 0 \). Now, let us define
\[
g(k) = \int_0^{\infty} \left( (x+2)^{1/p} - 1 \right)^k (x+2)^{(-1/p)(k+\alpha+1)} \, dx.
\]

Using integration by parts repeatedly, we can easily deduce that
\[
g(k) = \frac{p(R-1)^k R^{-(k+\alpha-p+1)}}{k+\alpha-p+1} + \frac{p^k(R-1)^{k-1} R^{-(k+\alpha-p)}}{(k+\alpha-p+1)(k+\alpha-p)}
\]
\[
+ \cdots + \frac{p k R^{-(\alpha-p+1)}}{(k+\alpha-p+1)(k+\alpha-p)(\alpha-p+1)}.
\]

Now, from the hypotheses that \( q \geq 1 \) and \( \alpha > 0 \), it follows that
THE ABEL-TYPE TRANSFORMATIONS INTO $\ell$

\[
g(k) \leq \frac{(R-1)^{k+\alpha}R^{-(k+\alpha)}}{k+\alpha} + \frac{k(R-1)^{k+\alpha-1}R^{-(k+\alpha-1)}}{(k+\alpha)(k+\alpha-1)}
+ \cdots + \frac{k(k-1)(k-2)\cdots R^{-(\alpha)}}{(k+\alpha)(k+\alpha-1)\cdots(\alpha)}
\leq \frac{S^{k+\alpha}}{k+\alpha} + \frac{kS^{k+\alpha-1}}{(k+\alpha)(k+\alpha-1)} + \cdots + \frac{k(k-1)(k-2)\cdots S^{\alpha}}{(k+\alpha)(k+\alpha-1)\cdots(\alpha)}.
\]

(3.24)

By writing the right-hand side of the preceding inequality using the summation notation, we obtain

\[
g(k) \leq \frac{S^{\alpha}}{\alpha(k+\alpha)} \sum_{i=0}^{k} \binom{i+\alpha-1}{i} S^i
\leq \frac{S^{\alpha}}{\alpha(k+\alpha)} \sum_{i=0}^{\infty} \binom{i+\alpha-1}{i} S^i
= \frac{S^{\alpha}}{\alpha(k+\alpha)} S^{-\alpha} = \frac{1}{\alpha(k+\alpha)}.
\]

(3.25)

Consequently, we have

\[
\sum_{n=0}^{\infty} |a_{nk}| \leq M \frac{(k+\alpha)}{\alpha(k+\alpha)} g(k) \leq \frac{M \frac{(k+\alpha)}{k}}{\alpha(k+\alpha)} = \frac{M}{\alpha}.
\]

(3.26)

Thus, by Knopp-Lorentz theorem [6], $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix.

Conversely, if $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then it follows, by Theorems 3 and 4, that $q \geq 1$.

\[\square\]

**Corollary 8.** Suppose that $t_n = 1 - (n+2)^{-q}$, $w_n = 1 - (n+2)^{-p}$ and $q < p$. Then $A_{\alpha,w}$ is an $\ell$-$\ell$ matrix whenever $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix.

**Proof.** The result follows immediately from Theorems 1 and 5.

**Corollary 9.** Suppose that $\alpha > 0$, $t_n = 1 - (n+2)^{-q}$, $w_n = 1 - (n+2)^{-p}$ and $(1/q) + (1/p) = 1$. Then both $A_{\alpha,t}$ and $A_{\alpha,w}$ are $\ell$-$\ell$ matrices.

**Proof.** The hypotheses imply that both $q$ and $p$ are greater than 1, and hence the corollary follows easily by Theorem 5.

**Theorem 6.** The following statements are equivalent:
1. $A_{\alpha,t}$ is a $G_w$-$\ell$ matrix;
2. $(1-t)^{\alpha+1} \in \ell$;
3. arcsin$(1-t)^{\alpha+1} \in \ell$;
4. $(1-t)^{\alpha+1}/(\sqrt{1-(1-t)^{2(\alpha+1)}}) \in \ell$;
5. $A_{\alpha,t}$ is a $G$-$\ell$ matrix.
**Proof.** We get \((1) \Rightarrow (2)\) by [9, Thm. 1.1] and \((2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)\) follow easily from the following basic inequality

\[
x < \arcsin x < \frac{x}{\sqrt{1 - x^2}}, \quad 0 < x < 1,
\]

and by [4, Thm. 1]. The assertion that \((5) \Rightarrow (1)\) follows immediately as \(G_w\) is a subset of \(G\).

**Corollary 10.** Suppose that \(t_n = 1 - (n + 2)^{-q}\). Then \(A_{\alpha,t}\) is a \(G-\ell\) matrix if and only if \(\alpha > (1 - q)/q\). For \(q = 1\), \(A_{\alpha,t}\) is a \(G-\ell\) matrix if and only if it is an \(\ell-\ell\) matrix.

**Proof.** The proof follows using Theorems 3 and 6.

**Theorem 7.** The following statements are equivalent:

1. \(A_{\alpha,t}\) is a \(G_w\)-\(G\) matrix;
2. \((1 - t)^{\alpha+1} \in G\);
3. \(\arcsin(1 - t)^{\alpha+1} \in G\);
4. \(A_{\alpha,t}\) is a \(G-\ell\) matrix.

**Proof.** \((1) \Rightarrow (2)\) follows by [9, Thm. 2.1] and \((2) \Rightarrow (3) \Rightarrow (4)\) follows easily from (3.27) and [4, Thm. 4]. The assertion that \((4) \Rightarrow (1)\) follows immediately as \(G_w\) is a subset of \(G\).

**Corollary 11.** If \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix, then it is a \(G-\ell\) matrix.

Our next few results suggest that the Abel-type matrix \(A_{\alpha,t}\) is \(\ell\)-stronger than the identity matrix (see [7, Def. 3]). The results indicate how large the sizes of \(\ell(A_{\alpha,t})\) and \(d(A_{\alpha,t})\) are.

**Theorem 8.** Suppose that \(-1 < \alpha \leq 0\), \(A_{\alpha,t}\) is an \(\ell-\ell\) matrix, and the series \(\sum_{k=0}^{\infty} x_k\) has bounded partial sums. Then it follows that \(x \in \ell(A_{\alpha,t})\).

**Proof.** Since, for \(-1 < \alpha \leq 0\), \((k+\alpha)_k\) is decreasing, the theorem is proved by following the same steps used in the proof of [7, Thm. 4].

**Remark 2.** Although the preceding theorem is stated for \(-1 < \alpha \leq 0\), the conclusion is also true for \(\alpha > 0\) for some sequences. This is demonstrated as follows: let \(x\) be the bounded sequence given by

\[
x_k = (-1)^k.
\]

Let \(Y\) be the \(A_{\alpha,t}\)-transform of the sequence \(x\). Then it follows that the sequence \(Y\) is given by

\[
Y_n = (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^k = (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} (-1)^k t_n^k = \frac{(1-t_n)^{\alpha+1}}{(1+t_n)^{\alpha+1}}
\]
which implies that

\[ Y_n < (1 - t_n)^{\alpha + 1}. \]  

(3.30)

Hence, if \( A_{\alpha,t} \) is an \( \ell \)-\( \ell \) matrix, then by Theorem 1, \( (1 - t)^{\alpha + 1} \in \ell \), and so \( x \in \ell(A_{\alpha,t}) \).

**Corollary 12.** Suppose that \(-1 < \alpha \leq 0\), \( A_{\alpha,t} \) is an \( \ell \)-\( \ell \) matrix. Then \( \ell(A_{\alpha,t}) \) contains the class of all sequences \( x \) such that \( \sum_{k=0}^{\infty} x_k \) is conditionally convergent.

**Remark 3.** In fact, we can give a further indication of the size of \( \ell(A_{\alpha,t}) \) by showing that if \( A_{\alpha,t} \) is an \( \ell \)-\( \ell \) matrix, then it also contains an unbounded sequence. To verify this, consider the sequence \( x \) given by

\[ x_k = (-1)^k \frac{k + \alpha + 1}{\alpha + 1}. \]

(3.31)

Let \( Y \) be the \( A_{\alpha,t} \)-transform of the sequence \( x \). Then we have

\[
Y_n = (1 - t_n)^{\alpha + 1} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^k
\]

\[= (1 - t_n)^{\alpha + 1} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} (-1)^k \frac{k + \alpha + 1}{\alpha + 1} t_n^k\]

(3.32)

and, consequently,

\[ Y_n < (1 - t_n)^{\alpha + 1}. \]

(3.33)

Hence, if \( A_{\alpha,t} \) is an \( \ell \)-\( \ell \) matrix, then by Theorem 1, \( (1 - t)^{\alpha + 1} \in \ell \), and so \( x \in \ell(A_{\alpha,t}) \). This example clearly indicates that \( A_{\alpha,t} \) is a rather strong method in the \( \ell \)-\( \ell \) setting for any \( \alpha > -1 \).

The \( \ell \)-\( \ell \) strength of the \( A_{\alpha,t} \) matrices can also be demonstrated by comparing them with the familiar Norland matrices \( (N_p) \) [3]. By using the same techniques used in the proof of [3, Thm. 8], we can show that the class of the \( A_{\alpha,t} \) matrix summability methods is \( \ell \)-stronger than the class of \( N_p \) matrix summability methods for some \( p \).

When discussing the \( \ell \)-\( \ell \) strength of \( A_{\alpha,t} \), or the size of \( \ell(A_{\alpha,t}) \), it is very important that we also determine the domain of \( A_{\alpha,t} \). The following proposition, which can be easily proved, gives a characterization of the domain of \( A_{\alpha,t} \).

**Proposition 1.** The complex number sequence \( x \) is in the domain of the matrix \( A_{\alpha,t} \) if and only if

\[ \limsup_{k} |x_k|^{1/k} \leq 1. \]  

(3.34)

**Remark 4.** Proposition 1 can be used as a powerful tool in making a comparison between the \( \ell \)-\( \ell \) strength of the \( A_{\alpha,t} \) matrices and some other matrices as shown by the following examples.

**Example 1.** The \( A_{\alpha,t} \) matrix is not \( \ell \)-stronger than the Borel matrix \( B[8, p. 53] \). To demonstrate this, consider the sequence \( x \) given by

\[ x_k = (-3)^k. \]

(3.35)
Then we have

\[(Bx)_n = \sum_{k=0}^{\infty} e^{-n} \frac{n^k}{k!} (-3)^k = e^{-4n}. \quad (3.36)\]

Thus, we have \(Bx \in \ell\) and hence \(x \in \ell(B)\), but by Proposition 1, \(x \notin \ell(A_{\alpha,t})\). Hence, \(A_{\alpha,t}\) is not \(\ell\)-stronger than \(B\).

**Example 2.** The \(A_{\alpha,t}\) matrix is not \(\ell\)-stronger than the familiar Euler-Knopp matrix \(E_r\) for \(r \in (0, 1)\). Also, \(E_r\) is not \(\ell\)-stronger than \(A_{\alpha,t}\). To demonstrate this, consider the sequence \(x\) defined by

\[x_k = (-q)^k \quad \text{and} \quad r = \frac{1}{q}, \quad (3.37)\]

where \(q > 1\). Let \(Y\) be the \(E_r\)-transform of the sequence \(x\). Then it is easy to see that the sequence \(Y\) is defined by

\[Y_n = \left(\frac{-1}{q}\right)^n. \quad (3.38)\]

Since \(q > 1\), we have \(Y \in \ell\) and hence \(x \in \ell(E_r)\), but \(x \notin \ell(A_{\alpha,t})\) by Proposition 1. Hence, \(A_{\alpha,t}\) is not \(\ell\)-stronger than \(E_r\). To show that \(E_r\) is not \(\ell\)-stronger than \(A_{\alpha,t}\), we let \(-1 < \alpha \leq 0\) and consider the sequence \(x\) that was constructed by Fridy in his example of [5, p. 424]. Here, we have \(x \notin \ell(E_r)\), but \(x \in \ell(A_{\alpha,t})\) by Theorem 8. Thus, \(E_r\) is not \(\ell\)-stronger than \(A_{\alpha,t}\).

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**References**


**Lemma:** Department of mathematics, Savannah state university, Savannah, Georgia 31404, USA