SOME APPLICATIONS OF A DIFFERENTIAL SUBORDINATION

YONG CHAN KIM and H. M. SRIVASTAVA

(Received 14 January 1998)

ABSTRACT. A number of interesting criteria were given by earlier workers for a normalized analytic function to be in the familiar class $S^*$ of starlike functions. The main object of the present paper is to extend and improve each of these earlier results. An application associated with an integral operator $F_c (c > -1)$ is also considered.

Keywords and phrases. Differential subordination, analytic functions, starlike functions, integral operator, Gauss hypergeometric function, Digamma function.

1991 Mathematics Subject Classification. Primary 30C45; Secondary 33B15, 33C05.

1. Introduction. Let $A(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\})$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $S^*$ be the class of starlike functions in $U$, defined by (cf., e.g., [2, 11])

$$S^* := \left\{ f(z) \in A(1) : \Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad (z \in U) \right\}.$$  \hspace{1cm} (1.2)

For analytic functions $g(z)$ and $h(z)$ with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$, $(z \in U)$, and $g(z) = h(w(z))$. We denote this subordination by $g(z) \prec h(z)$.

For a function $f(z)$ belonging to the class $A(1)$, Bernardi [1] defined the integral operator $F_c$ as follows:

$$(F_c f)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad (c > -1; \quad z \in U).$$ \hspace{1cm} (1.3)

We note that $F_c f \in A(n)$ if $f \in A(n)$. In particular, the operator $F_1$ was studied earlier by Libera [3]. (Also, see Owa and Srivastava [8, p. 126 et seq.]).

R. Singh and S. Singh [10] proved that if $f(z) \in A(1)$ and

$$\Re \{f'(z) + zf''(z)\} > -\frac{1}{4}, \quad (z \in U),$$ \hspace{1cm} (1.4)

then $f(z) \in S^*$.

Recently, Yi and Ding [12] improved the above-mentioned result of R. Singh and S. Singh [10] by showing that if $f(z) \in A(1)$ and

$$\Re \{f'(z) + zf''(z)\} > 1 - \frac{3}{4(1 - \log 2)^2 + 2} \approx -0.263, \quad (z \in U),$$ \hspace{1cm} (1.5)
then \( f(z) \in \mathcal{F}^* \).

Furthermore, Nunokawa and Thomas [6] proved that if \( f(z) \in \mathcal{A}(1) \) and
\[
\Re \{f'(z)\} > -0.0175\ldots, \quad (z \in \mathcal{U}),
\]
then \( \mathcal{F}_1 f \in \mathcal{F}^* \).

In this paper, we extend and improve each of these earlier results in [6, 12] and also consider an interesting application associated with the integral operator \( \mathcal{F}_c \).

2. Preliminary results. The following results are required in our investigation.

**Lemma 1** (Yi and Ding [12, Lem. 1]). Suppose that the function \( \phi: \mathbb{C} \times \mathbb{U} \to \mathbb{C} \) satisfies the condition \( \Re \{\phi(ix, y; z)\} \leq \delta \) for all real \( x \) and \( y \leq -\left(\frac{1}{2}\right)(1 + x^2) \) and all \( z \in \mathbb{U} \). If \( p(z) = 1 + p_1z + p_2z + \cdots \) is analytic in \( \mathbb{U} \) and
\[
\Re \{\phi(p(z), zp'(z); z)\} > \delta, \quad (z \in \mathbb{U}),
\]
then \( \Re \{p(z)\} > 0 \) in \( \mathbb{U} \).

**Lemma 2** (Owa and Nunokawa [7, Thm. 1]). Let \( p(z) \) be analytic in \( \mathbb{U} \) with
\[
p(0) = 1, \quad p'(0) = \cdots = p^{(n-1)}(0) = 0.
\]
If \( p(z) \) satisfies the inequality
\[
\Re \{p(z) + \alpha z p'(z)\} > \beta, \quad (z \in \mathbb{U}),
\]
then
\[
\Re \{p(z)\} > \beta + (1 - \beta) \left\{ 2 \int_0^1 \frac{d\rho}{1 + \rho^{n\Re(\alpha)}} - 1 \right\}, \quad (z \in \mathbb{U}),
\]
where \( \alpha \neq 0, \Re(\alpha) \geq 0, \) and \( \beta < 1 \).

**Lemma 3** (Owa and Nunokawa [7, Ex. 1]). Let \( \alpha > 0 \) and \( \beta < 1 \). If \( f(z) \in \mathcal{A}(n) \) satisfies the inequality
\[
\Re \{f'(z) + \alpha z f''(z)\} > \beta, \quad (z \in \mathbb{U}),
\]
then
\[
\Re \{f'(z)\} > \beta + (1 - \beta) \{2\delta(n, \alpha) - 1\}, \quad (z \in \mathbb{U}),
\]
where
\[
\delta(n, \alpha) = \int_0^1 \frac{d\rho}{1 + \rho^{n\alpha}}.
\]

Incidentally, the value of \( \delta(n, \alpha) \) in (2.7) can be expressed as the Gauss hypergeometric function
\[
_{2}F_1 \left( 1, \frac{1}{n\alpha}; 1 + \frac{1}{n\alpha}; -1 \right)
\]
which may also be rewritten in terms of the difference of two Digamma (or \( \psi \)-) functions
\[
\frac{1}{2n\alpha} \left[ \psi \left( \frac{1 + n\alpha}{2n\alpha} \right) - \psi \left( \frac{1}{2n\alpha} \right) \right] \quad \left( \psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \right).
\]

We also note that the inequality (2.5) is equivalent to the subordination given by
SOME APPLICATIONS OF A DIFFERENTIAL SUBORDINATION

\[ f'(z) + \alpha zf''(z) < \frac{1 + (1 - 2\beta)z}{1 - z}. \] (2.10)

3. Main results. The following theorem is a generalization of the main result of Yi and Ding [12].

**Theorem.** Let \( \delta(n, \alpha) \) be as defined in Lemma 3 and let \( \theta = 0.911621907, \alpha \geq 0.17418 \), and

\[ \alpha - \frac{(1 - \alpha)^2}{3\alpha} \tan^2 \theta < \frac{2\delta(n, \alpha) - 1}{[1 - \delta(n, \alpha)][2\delta(n, 1) - 1]} \] (3.1)

If \( f \in \mathcal{A}(n) \) satisfies the inequality

\[ \Re \{f'(z) + \alpha zf''(z)\} > 1 - \frac{2}{\alpha} + \left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right) \] (3.2)

then \( f(z) \in \mathcal{F}^* \).

**Proof.** Making use of Lemma 3 and the inequality (3.2), we obtain

\[ \Re \{f'(z)\} > \beta + (1 - \beta)\{2\delta(n, \alpha) - 1\} \]

\[ = 2\{\delta(n, \alpha) - 1\} \left[ \frac{2}{\alpha} + \left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right) \right] + 1 \]

\[ = \gamma, \quad (z \in \mathbb{U}), \] (3.3)

where

\[ \beta = 1 - \frac{2}{\alpha} + \left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right) \] (3.4)

Since \( \alpha \geq 0.17418 \) and

\[ \frac{1}{2} < \delta(n, \alpha) < 1, \quad (\alpha > 0; n \in \mathbb{N}), \] (3.5)

we have

\[ \frac{2}{\alpha} + 4\{1 - \delta(n, 1)\}\{1 - \delta(n, \alpha)\}\left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right) > 0. \] (3.6)

Hence, by (3.1), we find from (3.3) that

\[ 0 < \gamma < 1. \] (3.7)

If we put \( p(z) = z^{-1}f(z) \), then

\[ \Re \{f'(z)\} = \Re \{p(z) + zp'(z)\} > \gamma, \quad (z \in \mathbb{U}), \] (3.8)
which, in view of Lemma 2, implies that
\[ \Re \left\{ \frac{f(z)}{z} \right\} > \gamma + (1 - \gamma) \{2\delta(n,1) - 1\}, \quad (z \in \mathbb{U}). \] (3.9)

By using (3.5) and (3.7), we get
\[ \Re \left\{ \frac{f(z)}{z} \right\} > 0, \quad (z \in \mathbb{U}). \] (3.10)

Next, we let
\[ q(z) = zf'(z) f(z) \quad \text{and} \quad \lambda(z) = \frac{f(z)}{z}. \] (3.11)

Then
\[ \Re \{\lambda(z)\} > \gamma + (1 - \gamma) \{2\delta(n,1) - 1\}, \quad (z \in \mathbb{U}) \] (3.12)

and
\[ f'(z) + \alpha zf''(z) = \lambda(z) \left[ \alpha z q'(z) + (1 - \alpha) q(z) + \alpha [q(z)]^2 \right] \]
\[ = \phi(q(z), zq'(z); z), \] (3.13)

where \( \phi(u, v; z) = \lambda(z) \left[ \alpha u^2 + (1 - \alpha) u + \alpha v \right] \).

By setting \( \lambda(z) = a + bi \), we get
\[ \Re \{\phi(ix, y; z)\} \leq -\frac{1}{2} \left\{ 3\alpha x^2 + 2b(1 - \alpha)x + \alpha a \right\} \]
\[ \leq -\frac{a}{2} \left\{ \alpha - \frac{1}{3\alpha} (1 - \alpha)^2 \left( \frac{b}{a} \right)^2 \right\} \] (3.14)

for all real \( x \) and \( y \leq -\frac{1}{2}(1 + x^2) \). Since \( \Re\{f'(z)\} > 0 (z \in \mathbb{U}) \) implies that
\( \lambda(z) < L(z) := -1 - (2/\log(1 - z), \) we have \( \lambda(\mathbb{U}) \subset L(\mathbb{U}) \), where (see [9])
\[ L(\mathbb{U}) \subset \{ \omega : \Re(\omega) > 2\log 2 - 1 \} \cap \{ \omega : |\Im(\omega)| < \pi \} \cap \{ \omega : |\arg(\omega)| < \theta = 0.911621907 \}. \] (3.15)

By using (3.9) and (3.14), we obtain
\[ \Re \{\phi(ix, y; z)\} \leq -\frac{a}{2} \left\{ \alpha - \frac{(1 - \alpha)^2}{3\alpha} \tan^2 \theta \right\} \]
\[ \leq \beta, \quad (z \in \mathbb{U}). \] (3.16)

Hence, by Lemma 1, we get
\[ \Re \{q(z)\} = \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad (z \in \mathbb{U}). \] (3.17)

This evidently completes the proof of the theorem. \( \square \)

**Corollary 1.** Let \( \theta = 0.911621907, \alpha \geq 0.17418, \) and
\[ \alpha - \frac{(1 - \alpha)^2}{3\alpha} \tan^2 \theta < \frac{2\delta(1, \alpha) - 1}{1 - \delta(1, \alpha)} \left( 2\log 2 - 1 \right). \] (3.18)
If \( f \in \mathcal{A}(1) \) satisfies the inequality
\[
\Re \{ f'(z) + az f''(z) \} > 1 - \frac{2}{\alpha} \left( 1 - \frac{1}{\alpha} \right) \frac{\tan^2 \theta}{3} \left( 1 - \frac{1}{\alpha} \right) \left( 1 - \frac{1}{\alpha} \right) \left( 1 - \frac{1}{\alpha} \right) \left( 1 - \frac{1}{\alpha} \right) \left( 1 - \frac{1}{\alpha} \right) , \quad (z \in \mathbb{U}),
\] (3.19)
then \( f(z) \in \mathbb{P}^* \).

**Remark 1.** For \( \alpha = 1 \), Corollary 1 immediately yields the main result of Yi and Ding [12, Thm., p. 614].

**Remark 2.** A result of Ponnusamy [9, Thm. 4] can be obtained by taking \( \beta = 0 \) in the proof of our theorem.

It is not difficult to apply the definition (1.3) in order to show that
\[
f'(z) = (\mathcal{F}_c f)'(z) + \frac{1}{c+1} z(\mathcal{F}_c f)''(z).
\] (3.20)
Thus, by the theorem, we arrive at the following application:

**Corollary 2.** Let \( \theta = 0.911621907, -1 < c \leq 4.741187, \) and

\[
\frac{1}{c+1} - \frac{c^2}{3(c+1)} \tan^2 \theta < \frac{2\delta \left( n, \frac{1}{c+1} \right) - 1}{1 - \delta \left( 1, \frac{1}{c+1} \right) \{2\delta(n,1) - 1\}}.
\] (3.21)
If \( f \in \mathcal{A}(n) \) satisfies the inequality
\[
\Re \{ f'(z) \} > 1 - \frac{2(c+1) + \left( 1 - \frac{1}{3} c^2 \tan^2 \theta \right)}{2(c+1) + 4\{1 - \delta(n,1)\}\left( 1 - \frac{1}{3} c^2 \tan^2 \theta \right)\left( 1 - \frac{1}{3} c^2 \tan^2 \theta \right)}, \quad (z \in \mathbb{U}),
\] (3.22)
then \( \mathcal{F}_c f \in \mathbb{P}^* \), where \( \mathcal{F}_c \) is defined by (1.3).

By setting \( c = n = 1 \) in Corollary 2, we obtain Corollary 3 below, which shows that the constant \(-0.0175\) in the inequality (1.6) of Nunokawa and Thomas [6] can be reduced further.

**Corollary 3.** Let \( \theta = 0.911621907. \) If \( f \in \mathcal{A}(1) \) satisfies the inequality
\[
\Re \{ f'(z) \} > 1 - \frac{5 - (1/3) \tan^2 \theta}{4 + 8(1 - \log 2)^2\left( 1 - (1/3) \tan^2 \theta \right)} \approx -0.025311\ldots, \quad (z \in \mathbb{U}),
\] (3.23)
then \( \mathcal{F}_1 f \in \mathbb{P}^* \).

**Proof.** Since
\[
\frac{1}{2} - \frac{1}{6} \tan^2 \theta = 0.222356 \quad (\theta = 0.911621907) \quad \text{and} \quad \frac{3 - 4\log 2}{(2\log 2 - 1)^2} = 1.523967\ldots,
\] (3.24)
the proof of Corollary 3 is completed by setting \( c = n = 1 \) in Corollary 2. \qed
**Remark 3.** Several nonsharp results, obtained by various other authors (cf., e.g., [9]), correspond to the further special cases of Corollary 2 when $c = 0$ and $c = 1$.

**Acknowledgements.** The present investigation was initiated during the second-named author’s visit to Yeungnam University in December 1996. This work was partially supported by KOSEF and BSRI-97-1401 of Korea and by the Natural Sciences and Engineering Research Council of Canada under Grant OGP 0007353.

**References**


**Kim:** Department of Mathematics, College of Education, Yeungnam University, 214-1 Daedong, Gyongsan 712-749, Korea

**E-mail address:** kimyc@ynucc.yeungnam.ac.kr

**Srivastava:** Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada

**E-mail address:** hmsri@uvvm.uvic.ca