ON A CLASS OF UNIVALENT FUNCTIONS

DINGGONG YANG and JINLIN LIU

(Received 15 September 1997)

Abstract. We consider the class of univalent functions defined by the conditions \( f(z)/z \neq 0 \) and \( |(z/f(z))''| \leq \alpha, |z| < 1 \), where \( f(z) = z + \cdots \) is analytic in \( |z| < 1 \) and \( 0 < \alpha \leq 2 \).

Keywords and phrases. Univalent functions, subordination.

1991 Mathematics Subject Classification. 30C45.

1. Introduction. Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the unit disk \( E = \{ z : |z| < 1 \} \). A function \( f(z) \in A \) is said to be star-like in \( |z| < r (r \leq 1) \) if and only if it satisfies

\[
\Re \frac{zf'(z)}{f(z)} > 0, \quad (|z| < r).
\]

In [2], Nunokawa, Obradovic, and Owa proved the following theorem:

**Theorem A.** Let \( f(z) \in A \) with \( f(z) \neq 0 \) for \( 0 < |z| < 1 \) and let

\[
\left| \left( \frac{z}{f(z)} \right)'' \right| \leq 1, \quad (z \in E).
\]

Then \( f(z) \) is univalent in \( E \).

For \( 0 < \alpha \leq 2 \), let \( S(\alpha) \) denote the class of functions \( f(z) \in A \) which satisfy the conditions

\[
f(z) \neq 0 \quad \text{for} \quad 0 < |z| < 1
\]

and

\[
\left| \left( \frac{z}{f(z)} \right)'' \right| \leq \alpha, \quad (z \in E).
\]

In this paper, we give an extension of Theorem A and obtain some results for the class \( S(\alpha) \).

By virtue of a result due to Ozaki and Nunokawa [4], Obradovic et al. [3] considered a class of univalent functions.

2. A criterion for univalence

**Theorem 1.** Let \( f(z) \in A \) with \( f(z) \neq 0 \) for \( 0 < |z| < 1 \) and let \( g(z) \in A \) be bounded...
in $E$ and satisfy
\[ m = \inf \left\{ \left| \frac{g(z_1) - g(z_2)}{z_1 - z_2} \right| : z_1, z_2 \in E \right\} > 0. \quad (2.1) \]

If
\[ \left| \left( \frac{z}{f(z)} - \frac{z}{g(z)} \right)'' \right| \leq K, \quad (z \in E), \quad (2.2) \]

where
\[ K = \frac{2m}{M^2} \quad \text{and} \quad M = \sup \left\{ |g(z)| : z \in E \right\}, \quad (2.3) \]

then $f(z)$ is univalent in $E$.

**Proof.** If we put
\[ h(z) = \left( \frac{z}{f(z)} - \frac{z}{g(z)} \right)'' , \quad (2.4) \]

then the function $h(z)$ is analytic in $E$ and, by integration from 0 to $z$, we get
\[ \left( \frac{z}{f(z)} - \frac{z}{g(z)} \right)' = b_2 - a_2 + \int_0^z h(u) du \quad (2.5) \]

and
\[ \frac{z}{f(z)} - \frac{z}{g(z)} = (b_2 - a_2)z + \int_0^z dv \int_0^v h(u) du, \quad (2.6) \]

where $f(z) = z + a_2 z^2 + \cdots$ and $g(z) = z + b_2 z^2 + \cdots$.

Thus, we have
\[ f(z) = \frac{g(z)}{1 + (b_2 - a_2)g(z) + g(z)(\psi(z)/z)'}, \quad (2.7) \]

where
\[ \psi(z) = \int_0^z dv \int_0^v h(u) du. \quad (2.8) \]

Since
\[ \left( \frac{\psi(z)}{z} \right)' = \frac{1}{z^2} \int_0^z u \psi''(u) du = \frac{1}{z^2} \int_0^z uh(u) du, \quad (2.9) \]

from (2.2) and (2.4), we get
\[ \left| \left( \frac{\psi(z)}{z} \right)' \right| \leq \int_0^1 t|h(zt)| dt \leq \frac{K}{z}, \quad (2.10) \]

and so
\[ \left| \frac{\psi(z_2)}{z_2} - \frac{\psi(z_1)}{z_1} \right| = \left| \int_{z_1}^{z_2} \left( \frac{\psi(z)}{z} \right)' dz \right| \leq \frac{K}{z} |z_2 - z_1| \quad (2.11) \]

for $z_1, z_2 \in E$ and $z_1 \neq z_2$.

If $z_1 \neq z_2$ then $g(z_1) \neq g(z_2)$ and it follows, from (2.7) and (2.11), that
\[\left| f(z_1) - f(z_2) \right| = \frac{\left| g(z_1) - g(z_2) + g(z_1)g(z_2) \left( \frac{\psi(z_2)}{z_2} - \frac{\psi(z_1)}{z_1} \right) \right|}{\left| 1 + (b_2 - a_2)g(z_1) + g(z_1) \frac{\psi(z_1)}{z_1} \right| \left| 1 + (b_2 - a_2)g(z_2) + g(z_2) \frac{\psi(z_2)}{z_2} \right|} > \frac{|g(z_1) - g(z_2)| - M^2K\frac{|z_1 - z_2|}{2}}{1 + (b_2 - a_2)g(z_1) + g(z_1)\frac{\psi(z_1)}{z_1} \left| 1 + (b_2 - a_2)g(z_2) + g(z_2)\frac{\psi(z_2)}{z_2} \right|} \geq 0. \] (2.12)

Hence, \( f(z) \) is univalent in \( E \).

**COROLLARY 1.** Let \( f(z) \in A \) with \( f(z) \neq 0 \) for \( 0 < |z| < 1 \). If

\[ \left| \left( \frac{z}{f(z)} \right)^{''} \right| \leq 2, \quad (z \in E), \] (2.13)

then \( f(z) \) is univalent in \( E \). The bound 2 in (2.13) is best possible.

**PROOF.** Setting \( g(z) = z \) in Theorem 1, we conclude that \( f(z) \) is univalent in \( E \) for \( f(z) \) satisfying condition (2.13).

To show that the result is sharp, we consider

\[ f(z) = \frac{z}{(1 + z)^{2+\epsilon}}, \quad (\epsilon > 0). \] (2.14)

Note that

\[ \left| \left( \frac{z}{f(z)} \right)^{''} \right| = (2 + \epsilon)(1 + \epsilon)|1 + z|^\epsilon, \quad (z \in E) \] (2.15)

and \( f'(1/(1+\epsilon)) = 0 \). Hence, \( f(z) \) is not univalent in \( E \) and the proof is complete. \( \square \)

From Corollary 1, we easily get

**COROLLARY 2.** Let

\[ f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} n b_n z^n} \in A \] (2.16)

and

\[ \sum_{n=2}^{\infty} n(n-1)|b_n| \leq 2. \] (2.17)

Then \( f(z) \) is univalent in \( E \).

3. **The class \( S(\alpha) \).** According to Corollary 1, all the functions in \( S(\alpha)(0 < \alpha \leq 2) \) are univalent in \( E \). Let the functions \( f(z) \) and \( g(z) \) be analytic in \( E \). Then \( f(z) \) is said to be subordinate to \( g(z) \), written \( f(z) \prec g(z) \), if there exists a function \( w(z) \) analytic in \( E \), with \( w(0) = 0 \) and \( |w(z)| < 1 (z \in E) \), such that \( f(z) = g(w(z)) \) for \( z \in E \).

For our next results, we need the following.
**Lemma 1** [5]. Let \( f(z) \) and \( g(z) \) be analytic in \( E \) with \( f(0) = g(0) \). If \( h(z) = zg'(z) \) is star-like in \( E \) and \( zf'(z) < h(z) \), then
\[
f(z) < g(z) = g(0) + \int_0^z \frac{h(t)}{t} \, dt. \tag{3.1}
\]

**Theorem 2.** Let \( f(z) = z + a_2 z^2 + \cdots \in S(\alpha) \) with \( 0 < \alpha \leq 2 \). Then, for \( z \in E \),
\[
\left| \frac{z}{f(z)} - 1 \right| \leq |z| \left( |a_2| + \frac{\alpha}{2} |z| \right); \tag{3.2}
\]
\[
1 - |z| \left( |a_2| + \frac{\alpha}{2} |z| \right) \leq \text{Re} \frac{z}{f(z)} \leq 1 + |z| \left( |a_2| + \frac{\alpha}{2} |z| \right); \tag{3.3}
\]
\[
|f(z)| \geq \frac{|z|}{1 + |a_2| |z| + \frac{\alpha}{2} |z|^2}. \tag{3.4}
\]

Equalities in (3.2), (3.3), and (3.4) are attained if we take
\[
f(z) = \frac{z}{1 \pm az^2 + \frac{\alpha}{2} z^2} \in S(\alpha), \quad (0 \leq a \leq \sqrt{2\alpha}). \tag{3.5}
\]

**Proof.** In view of (1.5), we have
\[
z \left( \frac{z}{f(z)} \right)'' < \alpha z. \tag{3.6}
\]
Applying the lemma to (3.6), we find that
\[
\left( \frac{z}{f(z)} \right)' + a_2 < \alpha z. \tag{3.7}
\]
By using a result of Hallenbeck and Ruscheweyh [1, Thm. 1], (3.7) gives
\[
\frac{1}{z} \int_0^z \left[ \left( \frac{t}{f(t)} \right)' + a_2 \right] dt < \frac{\alpha}{2} z, \tag{3.8}
\]
i.e.,
\[
\frac{z}{f(z)} = 1 - a_2 z + \frac{\alpha}{2} z w(z), \tag{3.9}
\]
where \( w(z) \) is analytic in \( E \) and \(|w(z)| \leq |z|(|z| \in E)\) by Schwarz lemma.

Now, from (3.9), we can easily derive the inequalities (3.2), (3.3), and (3.4).

**Theorem 3.** Let \( f(z) \in S(\alpha) \) and have the form
\[
f(z) = z + a_3 z^3 + a_4 z^4 + \cdots. \tag{3.10}
\]

(a) If \( 2/\sqrt{5} \leq \alpha \leq 2 \), then \( f(z) \) is star-like in \(|z| < \sqrt{2/\alpha} \cdot 1/\sqrt{5}\);

(b) If \( \sqrt{3} - 1 \leq \alpha \leq 2 \), then \( \text{Re} f'(z) > 0 \) for \(|z| < \sqrt{((\sqrt{3} - 1)/\alpha)}\).

**Proof.** If we put
\[
p(z) = \frac{z^2 f'(z)}{f^2(z)} = 1 + p_2 z^2 + \cdots, \tag{3.11}
\]
then, by (1.5), we have

\[ zp'(z) = -z^2 \left( \frac{z}{f(z)} \right)'' < \alpha z, \]  

(3.12)

and it follows, from the lemma, that

\[ p(z) < 1 + \alpha z, \]  

(3.13)

which implies that

\[ \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq \alpha |z|^2, \quad (z \in E). \]  

(3.14)

(a) Let \( 2/\sqrt{5} \leq \alpha \leq 2 \) and

\[ |z| < r_1 = \sqrt{\frac{2}{\alpha} \cdot \frac{1}{\sqrt{5}}}. \]  

(3.15)

Then, by (3.14), we have

\[ \left| \arg \frac{z^2 f'(z)}{f^2(z)} \right| < \arcsin \frac{2}{\sqrt{5}}. \]  

(3.16)

Also, from (3.2) in Theorem 2 with \( a_2 = 0 \), we obtain

\[ \left| \frac{z}{f(z)} - 1 \right| < \frac{\alpha}{2} r_1^2, \]  

(3.17)

and so

\[ \left| \arg \frac{z}{f(z)} \right| < \arcsin \frac{1}{\sqrt{5}}. \]  

(3.18)

Therefore, it follows, from (3.16) and (3.18), that

\[ \left| \arg \frac{z^2 f'(z)}{f(z)} \right| \leq \left| \arg \frac{z^2 f'(z)}{f^2(z)} \right| + \left| \arg \frac{z}{f(z)} \right| < \arcsin \frac{2}{\sqrt{5}} + \arcsin \frac{1}{\sqrt{5}} = \frac{\pi}{2} \]  

(3.19)

for \( |z| < r_1 \). This proves that \( f(z) \) is star-like in \( |z| < r_1 \).

(b) Let \( \sqrt{3} - 1 \leq \alpha \leq 2 \) and

\[ |z| < r_2 = \sqrt{\frac{\sqrt{3} - 1}{\alpha}}. \]  

(3.20)

Then we have

\[ \left| \arg f'(z) \right| \leq \left| \arg \frac{z^2 f'(z)}{f^2(z)} \right| + 2 \left| \arg \frac{z}{f(z)} \right| < \arcsin (\alpha r_2^2) + 2 \arcsin \left( \frac{\alpha}{2} r_2^2 \right) \]  

\[ = \arcsin \left( \sqrt{3} - 1 \right) + 2 \arcsin \left( \frac{\sqrt{3} - 1}{2} \right) = \frac{\pi}{2}. \]  

(3.21)

Thus, \( \text{Re} f'(z) > 0 \) for \( |z| < r_2 \).

\[ \square \]

**Corollary 3.** Let \( f(z) \in S(\alpha) \) and have the form (3.10)

(a) If \( 0 < \alpha \leq 2/\sqrt{5} \), then \( f(z) \) is star-like in \( E \);

(b) If \( 0 < \alpha \leq \sqrt{3} - 1 \), then \( \text{Re} f'(z) > 0 \) for \( z \in E \).
References


Yang: Department of Mathematics, Suzhou University, Suzhou 215006, China

Liu: Water Conservancy College, Yangzhou University, Yangzhou 225009, China