RESEARCH NOTES

A NEW PROOF OF MONOTONICITY FOR EXTENDED MEAN VALUES

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(Received 1 January 1997 and in revised form 5 July 1998)

ABSTRACT. In this article, a new proof of monotonicity for extended mean values is given.

Keywords and phrases. Monotonicity, extended mean values, integral form, arithmetic mean, Tchebycheff’s integral inequality.

1991 Mathematics Subject Classification. Primary 26A48; Secondary 26D15.

1. Introduction. Stolarsky [14] first defined the extended mean values \( E(r, s; x, y) \) and proved that it is continuous on the domain \( \{(r, s; x, y) : r, s \in R, x, y > 0\} \) as follows

\[
E(r, s; x, y) = \left( \frac{r \cdot \frac{y^s - x^s}{y^r - x^r}}{\ln y - \ln x} \right)^{1/(s-r)}, \quad rs(r-s)(x-y) \neq 0; \tag{1.1}
\]

\[
E(r, 0; x, y) = \left( \frac{\frac{y^r - x^r}{\ln y - \ln x}}{1/r} \right)^{1/r}, \quad r(x-y) \neq 0; \tag{1.2}
\]

\[
E(r, r; x, y) = e^{-1/r} \left( \frac{\frac{x^r}{y^r}}{\ln y - \ln x} \right)^{1/(x^r-y^r)}, \quad r(x-y) \neq 0; \tag{1.3}
\]

\[
E(0, 0; x, y) = \sqrt{xy}, \quad x \neq y; \tag{1.4}
\]

\[
E(r, s; x, x) = x, \quad x = y. \tag{1.5}
\]

It is convenient to write \( E(r, s; x, y) = E(r, s) = E(x, y) = E \).

Several authors including Leach and Sholander [2, 3], Páles [6] and Yao and Cao [15] studied the basic properties, monotonicity and comparability of the mean values \( E \). Feng Qi [9] and in collaboration with Qiu-mig Luo [7] further investigated monotonicity of \( E \) from new viewpoints. Recently, Feng Qi [7] generalized the extended mean values and the weighted mean values [1, 4, 5] as a new concept of generalized weighted mean values with two parameters, and studied its monotonicity and other properties.

In this note, a new proof of monotonicity for extended mean values is given.

2. Lemmas. Let

\[
g = g(t) - g(t; x, y) = y^t - x^t/t, t \neq 0;
\]

\[
g(0; x, y) = \ln y - \ln x. \tag{2.1}
\]
It is easy to see that \( g \) can be expressed in integral form as

\[
g(t; x, y) = \int_x^y u^{t-1} du, \quad t \in \mathbb{R},
\]

and

\[
g^{(n)}(t) = \int_x^y (\ln u)^n u^{t-1} du, \quad t \in \mathbb{R}.
\]

Therefore, the extended mean values can be represented in terms of \( g \) by

\[
E(r, s; x, y) = \left( \frac{g(s; x, y)}{g(r; x, y)} \right)^{1/(s-r)}, \quad (r-s)(x-y) \neq 0;
\]

\[
E(r, r; x, y) = \exp \left( \frac{g'(r; x, y)}{g(r; x, y)} \right), \quad x-y \neq 0.
\]

Set \( F = F(r, s) = F(x, y) = F(r, s; x, y) = \ln E(r, s; x, y) \), then \( F \) also can be expressed as

\[
F(r, s; x, y) = \frac{1}{s-r} \int_s^r g(t; x, y) dt, \quad r-s \neq 0;
\]

\[
F(r, r; x, y) = \frac{g'(r; x, y)}{g(r; x, y)}.
\]

**Lemma 2.1.** Assume that the derivative \( f''(t) \) exists on an interval \( I \). If \( f(t) \) is an increasing or convex downward function respectively on \( I \), then the arithmetic mean of \( f(t) \),

\[
\phi(r, s) = \frac{1}{s-r} \int_r^s f(t) dt,
\]

\[
\phi(r, r) = f(r),
\]

is also increasing or convex downward respectively with \( r \) and \( s \) on \( I \).

**Proof.** Direct calculation yields

\[
\frac{\partial \phi(r, s)}{\partial s} = \frac{1}{(s-r)^2} \left[ (s-r)f(s) - \int_r^s f(t) dt \right],
\]

\[
\frac{\partial^2 \phi(r, s)}{\partial s^2} = \frac{(s-r)^2 f'(s) - 2(s-r)f(s) + 2 \int_r^s f(t) dt}{(s-r)^3} \equiv \frac{\phi'(r, s)}{(s-r)^3},
\]

\[
\frac{\partial \phi(r, s)}{\partial s} = (s-r)^2 f''(s).
\]

In the case of \( f'(t) \geq 0, \partial \phi(r, s)/\partial s \geq 0 \), thus, \( \phi(r, s) \) increases with \( r \) and \( s \), since \( \phi(r, s) = \phi(s, r) \).

In the case of \( f''(t) \geq 0, \phi(r, s) \) increases with \( s \). Since \( \phi(r, r) = 0 \), it is easy to see that \( \partial^2 \phi(r, s)/\partial s^2 \geq 0 \) holds. Therefore, \( \phi(r, s) \) is convex downward with respect to either \( r \) or \( s \), since \( \phi(r, s) = \phi(s, r) \).
Lemma 2.2. Let \( f, h : [a, b] \to \mathbb{R} \) be integrable functions, both increasing or both decreasing. Furthermore, let \( p : [a, b] \to \mathbb{R} \) be an integrable and nonnegative function. Then

\[
\int_a^b p(u) f(u) du \int_a^b p(u) h(u) du \leq \int_a^b p(u) du \int_a^b p(u) f(u) h(u) du.
\]  
(2.8)

If one of the functions of \( f \) or \( h \) is nonincreasing and the other nondecreasing, then the inequality in (2.8) is reversed.

The inequality (2.8) is called Tchebycheff’s integral inequality; for details, see [1, 4].

Lemma 2.3. Let \( i, j, k \in \mathbb{N} \), we have

\[
g^{(2(i+k)+1)}(t; x, y)g^{(2(j+k)+1)}(t; x, y) \leq g^{(2k)}(t; x, y)g^{(2(i+j+k)+1)}(t; x, y).
\]  
(2.9)

If \( x, y \geq 1 \), then

\[
g^{(i+k)}(t; x, y)g^{(j+k)}(t; x, y) \leq g^{(k)}(t; x, y)g^{(i+j+k)}(t; x, y).
\]  
(2.10)

If \( 0 < x, y \leq 1 \), then

\[
\begin{align*}
g^{(2(i+k)+1)}(t; x, y)g^{(2(j+k)+1)}(t; x, y) & \leq g^{(k)}(t; x, y)g^{(2(i+j+k)+1)}(t; x, y); \\
g^{(2(i+k)+1)}(t; x, y)g^{(2(j+k)+1)}(t; x, y) & \geq g^{(k)}(t; x, y)g^{(2(i+j+k)+1)}(t; x, y); \\
g^{(2(i+k)+1)}(t; x, y)g^{(2(j+k)+1)}(t; x, y) & \leq g^{(k)}(t; x, y)g^{(2(i+j+k)+1)}(t; x, y).
\end{align*}
\]  
(2.11)  
(2.12)  
(2.13)

Proof. By Tchebycheff’s integral inequality (2.8) applied to the functions \( p(u) = (\ln u)^{2k}u^{t-1}, f(u) = (\ln u)^{2i+1} \) and \( h(u) = (\ln u)^{2j+1} \) for \( i, j, k \in \mathbb{N}, u \in [x, y], t \in \mathbb{R} \), inequality (2.9) follows easily.

By the same arguments, inequalities (2.10), (2.11), (2.12), and (2.13) also follow from Tchebycheff’s integral inequality.

Lemma 2.4. The functions \( g^{(2(k+i)+1)}_t(t; x, y)/g^{(2k)}_t(t; x, y) \) are increasing with respect to \( t, x, \) and \( y \) for \( i \) and \( k \) being nonnegative integers.

Proof. By simple computation, we have

\[
\left( \frac{g^{(2(i+k)+1)}(t)}{g^{(2k)}(t)} \right)' = \frac{g^{(2(i+k+1))}(t)g^{(2k)}(t) - g^{(2(i+k)+1)}(t)g^{(2k+1)}(t)}{[g^{(2k)}(t)]^2}.
\]  
(2.14)

Combining (2.9) and (2.14), we conclude that the derivative of \( g^{(2(k+i)+1)}(t)/g^{(2k)}(t) \) with respect to \( t \) is nonnegative, and \( g^{(2(k+i)+1)}_t(t; x, y)/g^{(2k)}_t(t; x, y) \) increases with \( t \).

Differentiating directly, using the integral expression (2.3) of \( g \) and rearranging gives
\[
\frac{\partial}{\partial y} \left( \frac{g_t^{(2(k+i)+1)}(t;x,y)}{g_t^{(2k)}(t;x,y)} \right) = \\
= \frac{\partial}{\partial y} \left[ g_t^{(2(k+i)+1)}(t;x,y) \right] \frac{g_t^{(2k)}(t;x,y) - g_t^{(2(k+i)+1)}(t;x,y) \partial/\partial y \left[ g_t^{(2k)}(t;x,y) \right]}{\left[ g_t^{(2k)}(t;x,y) \right]^2} = \frac{y^{t-1} (\ln y)^{2k}}{\left[ g_t^{(2k)}(t;x,y) \right]^2} \left[ (\ln y)^{2i+1} \int_x^y (\ln u)^{2k} u^{t-1} du - \int_x^y (\ln u)^{2(i+k)+1} u^{t-1} du \right] \geq 0.
\]

(2.15)

Therefore, the desired monotonicity with respect to both \( x \) and \( y \) follows, for the involved functions are symmetric in \( x \) and \( y \). This completes the proof.

\[\square\]

3. Proof of monotonicity

**Theorem 3.1.** The extended mean values \( E(r,s;x,y) \) are increasing with respect to both \( r \) and \( s \), or to both \( x \) and \( y \).

**Proof.** This is a simple consequence of Lemma 2.1 and Lemma 2.3 in combination with its integral forms (2.4) and (2.5) of \( E(r,s;x,y) \).

**Remark 1.** It may be pointed out that the method used in this paper could yield more general results (see \([4, 12]\), and so on).

**Acknowledgement.** The first author was partially supported by NSF grant 974050400 of Henan Province, China.

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