ON $\theta$-GENERALIZED CLOSED SETS

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ABSTRACT. The aim of this paper is to study the class of $\theta$-generalized closed sets, which is properly placed between the classes of generalized closed and $\theta$-closed sets. Furthermore, generalized $\Lambda$-sets [16] are extended to $\theta$-generalized $\Lambda$-sets and $R_0$, $T_{1/2}$, and $T_1$-spaces are characterized. The relations with other notions directly or indirectly connected with generalized closed sets are investigated. The notion of TGO-connectedness is introduced.

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1. Introduction. The first step of generalizing closed sets was done by Levine in 1970 [15]. He defined a set $A$ to be generalized closed if its closure belongs to every open superset of $A$ and introduced the notion of $T_{1/2}$-spaces, which is properly placed between $T_0$-spaces and $T_1$-spaces. Dunham [10] proved that a topological space is $T_{1/2}$ if and only if every singleton is open or closed. In [13], Khalimsky, Kopperman, and Meyer proved that the digital line is a typical example of a $T_{1/2}$-space.

Ever since, general topologists extended the study of generalized closed sets on the basis of generalized open sets: regular open, $\alpha$-open [20], semi-open [14], semi-preopen [1], preopen [19], $\theta$-open [26], $\delta$-open [26], etc.

Extensive research on generalizing closedness was done in recent years as the notions of semi-generalized closed, generalized semi-closed, generalized $\alpha$-closed, $\alpha$-generalized closed, generalized semi-preclosed, regular generalized closed, $\gamma$-$g$-closed and $(\gamma, \gamma')$-$g$-closed sets were investigated [2, 3, 6, 7, 11, 18, 17, 22, 23, 24, 25].

Recently, in [8], Ganster and the first author of this paper defined $\delta$-generalized closed sets and introduced the notion of $T_{3/4}$-spaces, which is properly placed between $T_1$-spaces and $T_{1/2}$-spaces. They proved that the digital line is $T_{3/4}$.

The aim of this paper is to continue the study of generalized closed sets, this time via the $\theta$-closure operator defined in [26] and characterize $T_{1/2}$-spaces and $T_1$-spaces in terms of $\theta$-generalized closed sets. Via $\theta$-closure operator, we extend the class of generalized $\Lambda$-sets to the class of $\theta$-generalized $\Lambda$-sets and study some new characterizations of $R_0$-spaces and $T_1$-spaces.

2. Preliminaries concerning generalized closed sets. Throughout this paper, we consider spaces on which no separation axioms are assumed unless explicitly stated. The topology of a given space $X$ is denoted by $\tau$ and $(X, \tau)$ is replaced by $X$ if there is no chance for confusion. For $A \subseteq X$, the closure and the interior of $A$ in $X$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. Sometimes, when there is no chance for
confusion, $\overline{A}$ stands for $\text{Cl}(A)$. The $\theta$-interior [26] of a subset $A$ of $X$ is the union of all open sets of $X$ whose closures are contained in $A$, and is denoted by $\text{Int}_\theta(A)$. The subset $A$ is called $\theta$-open [26] if $A = \text{Int}_\theta(A)$. The complement of a $\theta$-open set is called $\theta$-closed. Alternatively, a set $A \subset (X, \tau)$ is called $\theta$-closed [26] if $\text{Cl}_\theta(A) = \{x \in X : \overline{U \cap A} \neq \emptyset, U \in \tau \text{ and } x \in U\}$. The family of all $\theta$-open sets forms a topology on $X$ and is denoted by $\tau_\theta$. We use the name CO-set for sets whose closure is open.

**Observation 2.1.**

(i) If $A$ is preopen, then $\text{Cl}_\alpha(A) = \text{Cl}(A) = \text{Cl}_\theta(A)$.

(ii) Every CO-set is preopen.

(iii) Every dense subset is a CO-set.

(iv) Every subset of a space $(X, \tau)$ is a CO-set if and only if $(X, \tau)$ is locally indiscrete.

**Definition 1.** A subset $A$ of a space $(X, \tau)$ is called

1. a general closed set (= g-closed) [15] if $A \subseteq U$ and $U \in \tau$ implies that $\overline{A} \subseteq U$,
2. a semi-generalized closed set (= sg-closed) [4] if $A \subseteq U$ and $U$ is semi-open implies that $\text{Cl}_\alpha(A) \subseteq U$,
3. a generalized $\alpha$-closed set (= $g\alpha$-closed) [17] if $A \subseteq U$ and $U$ is $\alpha$-open implies that $\text{Cl}_\alpha(A) \subseteq U$,
4. a generalized semi-closed set (= gs-closed) [2] if $A \subseteq U$ and $U \in \tau$ implies that $\text{Cl}(A) \subseteq U$,
5. an $\alpha$-generalized closed set (= $\alpha$ g-closed) [18] if $A \subseteq U$ and $U \in \tau$ implies that $\text{Cl}_\alpha(A) \subseteq U$,
6. a generalized semi-preclosed set (= gsp-closed) [7] if $A \subseteq U$ and $U \in \tau$ implies that $\text{spCl}(A) \subseteq U$,
7. a regular generalized closed set (= r-g-closed) [23] if $A \subseteq U$ and $U$ is regular open implies that $\overline{A} \subseteq U$.

**Definition 2.** A topological space $(X, \tau)$ is called

1. $R_0$-space [5] if the closures of every two different points are either disjoint or coincide,
2. $R_1$-space [5] if every two different points, with distinct closures, have disjoint neighborhoods,
3. $T_{1/2}$-space [15] if every g-closed set is closed, (= every singleton is open or closed [10]),
4. $kc$-space [27] if every compact set is closed.

**Definition 3.** Recall that a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

1. $g$-continuous [3] if $f^{-1}(V)$ is g-closed in $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$,
2. semi-continuous [14] if $f^{-1}(V)$ is semi-open in $(X, \tau)$ for every open set $V$ of $(Y, \sigma)$,
3. strongly $\theta$-continuous [21] if, for each $x \in X$ and each open set $V$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq V$.

3. Basic properties of $\theta$-generalized closed sets

**Definition 4.** A subset $A$ of a topological space $(X, \tau)$ is called $\theta$-generalized closed (= $\theta$-g-closed) if $\text{Cl}_\theta(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. 
We denote the family of all \( \theta \)-generalized closed subsets of a space \((X, \tau)\) by \( \text{TGC}(X, \tau) \).

The next two results together with the examples following them show that the class of \( \theta \)-generalized closed sets is properly placed between the classes of g-closed and \( \theta \)-closed sets.

**Observation 3.1.** Every \( \theta \)-closed set is \( \theta \)-generalized closed.

**Example 3.2.** Let \( X = \{a, b, c\} \) and let \( \tau = \{\emptyset, \{a, b\}, X\} \). Set \( A = \{a, c\} \). Since the only open superset of \( A \) is \( X \), \( A \) is clearly \( \theta \)-generalized closed. But it is easy to see that \( A \) is not \( \theta \)-closed. In fact, it is not even semi-closed since its complement \( \{b\} \) has empty interior.

**Observation 3.3.** Every \( \theta \)-generalized closed set is g-closed and hence \( \alpha \) g-closed, gs-closed, and r-g-closed.

**Example 3.4.** Let \( X = \{a, b, c\} \) and let \( \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\} \). Set \( A = \{c\} \). Clearly, \( A \) is closed and hence g-closed. Next, set \( U = \{a, c\} \). Note that \( X = \text{Cl}_\theta(A) \notin U \in \tau \). Thus, \( A \) is not \( \theta \)-generalized closed.

The following diagram is an enlargement of a Diagram from [7].

```
θ-closed ↘ θ-g-closed
   ↓
closed set ↘ g-closed set ↘ α-g-closed
      ↓
α-closed set ↘ gα-closed set ↘ gs-closed set ↘ r-g-closed
      ↓
semi-closed set ↘ sg-closed set ↘ gsp-closed set
      ↓
semi-preclosed set
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**Observation 3.5.** Let \((X, \tau)\) be a regular space (not necessarily even \( T_0 \)). Then a subset \( A \) of \( X \) is \( \theta \)-generalized closed if and only if \( A \) is generalized closed.

**Lemma 3.6** [12, Thm. 3.1(d), Thm. 3.6(d)]. For a space \((X, \tau)\), the following conditions are equivalent

1. \( X \) is an \( R_1 \)-space;
2. for each \( x \in X \), \( \text{Cl}\{x\} = \text{Cl}_\theta\{x\} \);
3. for each compact set \( A \subseteq X \), \( \text{Cl}(A) = \text{Cl}_\theta(A) \).

**Proposition 3.7.** If \((X, \tau)\) is \( R_1 \), then a compact subset \( K \) of \( X \) is g-closed if and only if \( K \) is \( \theta \)-g-closed.

**Proposition 3.8.** Let \( A \) be a preopen subset of a topological space \((X, \tau)\). Then the
following conditions are equivalent
(1) $A$ is $\theta$-$g$-closed;
(2) $A$ is $g$-closed;
(3) $A$ is $\alpha g$-closed.

**Proof.** Follows easily from Observation 2.1(i) (note that a preopen $g$-closed set is a CO-set).

**Lemma 3.9.** If $A$ and $B$ are subsets of a topological space $(X, \tau)$, then $\text{Cl}_\theta(A \cup B) = \text{Cl}_\theta(A) \cup \text{Cl}_\theta(B)$ and $\text{Cl}_\theta(A \cap B) \subseteq \text{Cl}_\theta(A) \cap \text{Cl}_\theta(B)$.

**Proposition 3.10.** (i) A finite union of $\theta$-$g$-closed sets is always a $\theta$-$g$-closed set.
(ii) A countable union of $\theta$-$g$-closed sets need not be a $\theta$-$g$-closed set.
(iii) A finite intersection of $\theta$-$g$-closed sets may fail to be a $\theta$-$g$-closed set.

**Proof.** (i) Let $A, B \in \text{TGC}(X)$. Let $U \in \tau$ such that $A \cup B \subseteq U$. By Lemma 3.9, $\text{Cl}_\theta(A \cup B) = \text{Cl}_\theta(A) \cup \text{Cl}_\theta(B) \subseteq U \cup U = U$ since $A$ and $B$ are $\theta$-$g$-closed. Hence, $A \cup B$ is $\theta$-$g$-closed.

(ii) Let $X$ be the real line with the usual topology. Since $X$ is regular, by Observation 3.5, every singleton in $X$ is $\theta$-$g$-closed. Set $A = \bigcup_{i=2}^{n} \{1/i\}$. Clearly, $A$ is a countable union of $\theta$-generalized closed sets but $A$ is not $\theta$-generalized closed since $A \in (0,1)$ and $0 \in \text{Cl}_\theta(A)$.

(iii) Let $X = \{a, b, c, d, e\}$ and let $\tau = \{\emptyset, \{a, b\}, \{c\}, \{a, b, c\}, X\}$. Set $A = \{a, c, d\}$ and $B = \{b, c, e\}$. Clearly, $A$ and $B$ are $\theta$-generalized closed sets since $X$ is their only open superset. But $C = \{c\} = A \cap B$ is not $\theta$-generalized closed since $C \subseteq \{c\} \in \tau$ and $\text{Cl}_\theta(C) = \{c, d, e\} \not\subseteq \{c\}$.

**Proposition 3.11.** The intersection of a $\theta$-generalized closed set and a $\theta$-closed set is always $\theta$-generalized closed.

**Proof.** Let $A$ be $\theta$-generalized closed and let $F$ be $\theta$-closed. Let $U$ be an open set such that $A \cap F \subseteq U$. Set $G = X \setminus F$. Then $A \subseteq U \cup G$. Since $G$ is $\theta$-open, $U \cup G$ is open and since $A$ is $\theta$-generalized closed, $\text{Cl}_\theta(A) \subseteq U \cup G$. By Lemma 3.9, $\text{Cl}_\theta(A \cap F) \subseteq \text{Cl}_\theta(A) \cap \text{Cl}_\theta(F) = \text{Cl}_\theta(A) \cap (U \cup G) \cap F = (U \cap F) \cup (G \cap F) = (U \cap F) \cup \emptyset \subseteq U$.

**Proposition 3.12.** Let $B \subseteq H \subseteq (X, \tau)$ and $(\text{Cl}_\theta)_{H}(B)$ denote the $\theta$-closure of $B$ in the subspace $(H, \tau \mid H)$. Then
(i) $(\text{Cl}_\theta)_{H}(B) \subseteq \text{Cl}_\theta(B) \cap H$ holds.
(ii) If $H$ is open in $(X, \tau)$, then $(\text{Cl}_\theta)_{H}(B) \supseteq \text{Cl}_\theta(B) \cap H$ holds.

**Theorem 3.13.** Let $B \subseteq H \subseteq (X, \tau)$.
(i) If $B$ is $\theta$-$g$-closed relative to $H$ (i.e., $B \in \text{TGC}(H, \tau \mid H)$), $H \in \text{TGC}(X)$, and $H \in \tau$, then $B \in \text{TGC}(X)$.
(ii) If $B$ is $\theta$-$g$-closed in $(X, \tau)$, then $B$ is $\theta$-$g$-closed relative to $H$ (i.e., $B \in \text{TGC}(H, \tau \mid H)$).

**Proof.** (i) Let $B \subseteq U$, where $U \in \tau$. Then $B \subseteq H \cap U$ and, moreover, $(\text{Cl}_\theta)_{H}(B) \subseteq H \cap U$ due to assumption. By Proposition 3.12(ii), $H \cap \text{Cl}_\theta(B) \subseteq H \cap U \subseteq U$. Using the last inclusion, it follows that $H \subseteq H \cup (X \setminus \text{Cl}_\theta(B)) = (H \cap \text{Cl}_\theta(B)) \cup (X \setminus \text{Cl}_\theta(B)) \subseteq U \cup (X \setminus \text{Cl}_\theta(B))$. Since $\text{Cl}_\theta(B)$ is a closed set, $U \cup (X \setminus \text{Cl}_\theta(B))$ is open and thus since $H \in \text{TGC}(X)$, $\text{Cl}_\theta(H) \subseteq U \cup (X \setminus \text{Cl}_\theta(B))$. Now, $\text{Cl}_\theta(B) \subseteq \text{Cl}_\theta(H) \subseteq U \cup (X \setminus \text{Cl}_\theta(B))$. From the
last inclusion, it follows that $\text{Cl}_B(B) \subseteq U$ or, equivalently, $B \in \text{TGC}(X)$.

(ii) Let $V$ be an open set of $(H, \tau \mid H)$ such that $B \subseteq V$. Then there exists an open set $G \in \tau$ such that $G \cap H = V$. Since $B \subseteq G \cap H \subseteq G$ and $B \in \text{TGC}(X)$, $\text{Cl}_B(B) \subseteq G$. By Proposition 3.12(ii), $(\text{Cl}_B(B)) \subseteq \text{Cl}_B(B) \cap H \subseteq G \cap H \subseteq V$. Therefore, $B$ is $g$-closed relative to $H$.

**Example 3.14.** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$. Then $\{\emptyset, X\}$ is the set of all $\theta$-closed sets of $(X, \tau)$ and $\text{TGC}(X, \tau) = \{\emptyset, \{b, c\}, \{b, d\}, \{b, c, d\}, \{a, b, c\}, X\}$. Let $H = \{b, c\}$ be a set of $X$. Then, $\tau \mid H = \{\emptyset, \{b\}, \{c, d\}, H\}$. Note that $\{\emptyset, \{b\}, \{c, d\}, H\}$ is the set of all $\theta$-closed sets of $(H, \tau \mid H)$ and $\text{TGC}(H, \tau \mid H) = \emptyset$. The subset $\{b\}$ of $H$ is $g$-closed relative to $H$ and $H$ is not open (i.e., $\{b\} \in \text{TGC}(H, \tau \mid H)$, $H \notin \tau$) and $H \in \text{TGC}(X, \tau)$. However, $\{b\} \notin \text{TGC}(X, \tau)$.

**Example 3.15.** Let $(X, \tau)$ be the space in the example above. Set $H = \{a, c, d\}$. Clearly, $H$ is open in $(X, \tau)$ and $H$ is not $\theta$-generalized closed in $(X, \tau)$. But $B = \{a, c\}$ is $\theta$-generalized closed relative to $H$. However, $B$ is not $\theta$-generalized closed in $(X, \tau)$.

4. Characterizations of $T_{1/2}$-spaces, $T_1$-spaces and $R_0$-spaces

**Theorem 4.1.** A space $(X, \tau)$ is a $T_{1/2}$-space if and only if every $\theta$-generalized closed set is closed.

**Proof.**

**Necessity.** Let $A \subseteq X$ be $\theta$-generalized closed. By Observation 3.3, $A$ is $g$-closed. Since $X$ is a $T_{1/2}$-space, $A$ is closed.

**Sufficiency.** Let $x \in X$. If $\{x\}$ is not closed, then $B = X \setminus \{x\}$ is not open and thus the only superset of $B$ is $X$. Trivially, $B$ is $\theta$-generalized closed. By (2), $B$ is closed or, equivalently, $\{x\}$ is open. Thus, every singleton in $X$ is open or closed. Hence, in the notion of [6, Thm. 6.2(i)], $X$ is a $T_{1/2}$-space.

**Lemma 4.2.** Let $A \subseteq (X, \tau)$ be $\theta$-generalized closed. Then $\text{Cl}_A(A) \setminus A$ does not contain a nonempty closed set.

**Theorem 4.3.** A space $(X, \tau)$ is a $T_1$-space if and only if every $\theta$-generalized closed set is $\theta$-closed.

**Proof.**

**Necessity.** Let $A \subseteq X$ be $\theta$-generalized closed and let $x \in \text{Cl}_A(A)$. Since $X$ is a $T_1$, $\{x\}$ is closed and thus by Lemma 4.2, $x \notin \text{Cl}_A(A) \setminus A$. Since $x \in \text{Cl}_A(A)$, then $x \in A$. This shows that $\text{Cl}_A(A) \subseteq A$ or, equivalently, that $A$ is $\theta$-closed.

**Sufficiency.** Let $x \in X$. Assume that $\{x\}$ is not closed. Then $B = X \setminus \{x\}$ is not open and, trivially, $B$ is $\theta$-generalized closed since the only open superset of $B$ is $X$ itself. By (2), $B$ is $\theta$-closed and thus $\{x\}$ is $\theta$-open. Since a singleton is $\theta$-open if and only if it is clopen, $\{x\}$ is clopen. 

The notion of a $\Lambda$-set and a generalized $\Lambda$-set in a topological space was introduced in [16]. By definition, a subset $A$ of a topological space $(X, \tau)$ is called a $\Lambda$-set [16] if $A = A^\Lambda$, where $A^\Lambda = \cap \{U : U \supset A, U \in \tau\}$. Recall that $A$ is called a generalized $\Lambda$-set [16] if $A^\Lambda \subseteq F$, whenever $A \subseteq F$ and $F$ is $\tau$-closed.
**Definition 5.** (i) For a subset $A$ of $(X, \tau)$, we define $A^\Lambda_\theta$ as follows

$$A^\Lambda_\theta = \{ x \in X : \text{Cl}_{\theta} \{ x \} \cap A \neq \emptyset \}.$$ 

In [12], $A^\Lambda_\theta$ is denoted by ker$\theta A$.

(ii) A subset $A$ of $(X, \tau)$ is called $\theta$-generalized $\Lambda$-set ($= \theta$-g-$\Lambda$-set) if $A^\Lambda_\theta \subseteq F$, whenever $A \subseteq F$ and $F$ is closed in $(X, \tau)$.

**Observation 4.4.** (i) Every $G_\delta$-set is a $\Lambda$-set.

(ii) [12, Lem. 3.5(a)]. For any set $A \subseteq X$, $A \subseteq A^\Lambda_\theta \subseteq \text{Cl}_{\theta}(A)$.

(iii) Every $\theta$-closed set is a $\Lambda$-set.

(iv) Every $g$-closed $\Lambda$-set is closed.

(v) Every $\theta$-generalized $\Lambda$-set is a generalized $\Lambda$-set.

**Remark 4.5.** (i) A $\Lambda$-set need not be $\theta$-closed. Any singleton of an infinite space with the cofinite topology is a $\Lambda$-set (since the space is $T_1$) but none of the singletons is $\theta$-closed.

(ii) A closed set need not be a $\Lambda$-set. In the Sierpinski space $(X = \{a, b\}, \tau = \{\emptyset, \{a\}, X\})$, the set $B = \{b\}$ is closed but $B$ is not a $\Lambda$-set. However, in [16, Prop. 3.8], it was shown that in a topological space $(X, \tau)$, every subset of $X$ is a generalized $\Lambda$-set if and only if every closed set is a $\Lambda$-set.

(iii) A generalized $\Lambda$-set need not be $\theta$-generalized $\Lambda$-set. In an infinite cofinite space $X$, as mentioned in Remark 4.5, every singleton is a $\Lambda$-set and, hence, a generalized $\Lambda$-set but none of the singletons is a $\theta$-generalized $\Lambda$-set since the $\theta$-closure of every singleton is $X$.

In [16], it was proved that in $T_1$-spaces, every set is a $\Lambda$-set. Note that the converse is also true.

**Proposition 4.6.** (i) A topological space $(X, \tau)$ is a $T_1$-space if and only if every subset of $X$ is a $\Lambda$-set.

(ii) A topological space $(X, \tau)$ is an $R_0$-space if and only if every singleton of $X$ is a generalized $\Lambda$-set.

**Proof.** (i) Obvious.

(ii) In [9], Dube showed that a space is $R_0$ if and only if, for each closed set $A$, $A = A^\Lambda$. Thus, if $X$ is $R_0$, then for each singleton $\{x\}$ and each closed set $F$ containing $x$, we have $\{x\} \subseteq \{x\}^\Lambda \subseteq F^\Lambda = F$. So, $\{x\}$ is a generalized $\Lambda$-set. For the reverse assume that $F \subseteq X$ is closed. For each $x \in F$, by assumption, $\{x\}^\Lambda \subseteq F$. Thus, $F^\Lambda = \bigcup_{x \in F} \{x\}^\Lambda \subseteq F$ according to [16, condition (2.5)]. This shows that $F = F^\Lambda$.

**Observation 4.7.** (i) A subset $A$ of an $R_1$-space $X$ is generalized $\Lambda$-set if and only if $A$ is $\theta$-generalized $\Lambda$-set.

(ii) In Hausdorff spaces, every subset is a $\theta$-generalized $\Lambda$-set.

(iii) A topological space $X$ is Hausdorff if and only if $X$ is a $kc$-space and every closed set of $X$ is a $\theta$-generalized $\Lambda$-set.

5. $\theta$-g-continuous and $\theta$-g-irresolute functions

**Definition 6.** A function $f : (X, \tau) \to (Y, \sigma)$ is called
1. **\( \theta \)-continuous** if \( f^{-1}(V) \) is \( \theta \)-g-closed in \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

\( \theta \)-g-closed sets are not necessarily \( \theta \)-continuous.

2. **\( \theta \)-irresolute** if \( f^{-1}(V) \) is \( \theta \)-g-closed in \((X, \tau)\) for every \( \theta \)-g-closed set \( V \) of \((Y, \sigma)\).

**Observation 5.1.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is strongly \( \theta \)-continuous, then \( f \) is \( \theta \)-g-continuous.

**Example 5.2.** Let \((X, \tau)\) be the space in Example 3.2. Let \( \sigma = \{ \emptyset, \{ b \}, X \}\). Let \( f : (X, \tau) \rightarrow (X, \sigma) \) be the identity function. Clearly, in the notion of Example 3.2, \( f \) is \( \theta \)-g-continuous but not strictly \( \theta \)-continuous, not even semi-continuous.

**Observation 5.3.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be \( \theta \)-g-continuous. Then \( f \) is g-continuous but not conversely.

**Example 5.4.** Let \((X, \tau)\) be the space in Example 3.4. Let \( \sigma = \{ \emptyset, \{ a, b \}, X \}\). Let \( f : (X, \tau) \rightarrow (X, \sigma) \) be the identity function. Clearly, \( f \) is continuous and hence g-continuous but as shown in Example 3.4, \( A = \{ c \} \notin \text{TGC}(X, \tau) \) and hence \( f \) is not \( \theta \)-g-continuous.

Example 5.2 and Example 5.4 also show that continuity and \( \theta \)-g-continuity are independent concepts. Thus, we have the following implications and none of them is reversible.

\[
\begin{array}{ccc}
\theta\text{-g-continuous} & \rightarrow & \text{Strongly } \theta\text{-continuous} & \rightarrow & \text{g-continuous} \\
\downarrow & & & & \downarrow \\
\text{continuous} & & & & \text{continuous}
\end{array}
\]

**Example 5.5.** Let \( f \) be the function in Example 5.2. Let \( \nu = \{ \emptyset, \{ c \}, X \}\). Let \( g : (X, \sigma) \rightarrow (X, \nu) \) be the identity function. It is easily observed that \( g \) is also \( \theta \)-generalized continuous. But the composition function \( g \circ f : (X, \tau) \rightarrow (X, \nu) \) is not \( \theta \)-g-continuous since \( \{ a, b \} \notin \text{TGC}(X, \tau) \).

**Theorem 5.6.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is bijective, open and \( \theta \)-generalized continuous, then \( f \) is \( \theta \)-g-irresolute.

**Proof.** Let \( V \in \text{TGC}(Y) \) and let \( f^{-1}(V) \subseteq O \), where \( O \in \tau \). Clearly, \( V \subseteq f(O) \). Since \( f(O) \in \sigma \) and since \( V \in \text{TGC}(Y) \), \( \text{Cl}_\theta(V) \subseteq f(O) \) and thus \( f^{-1}(\text{Cl}_\theta(V)) \subseteq O \). Since \( f \) is \( \theta \)-generalized continuous and since \( \text{Cl}_\theta(V) \) is closed in \( Y \), \( \text{Cl}_\theta(f^{-1}(\text{Cl}_\theta(V))) \subseteq O \) and hence \( \text{Cl}_\theta(f^{-1}(V)) \subseteq O \). Therefore, \( f^{-1}(V) \in \text{TGC}(X) \). Hence, \( f \) is \( \theta \)-g-irresolute.

**Definition 7.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called \( \theta \)-generalized closed if, for every closed set \( F \) of \((X, \tau)\), \( f(F) \) is \( \theta \)-g-closed in \((Y, \sigma)\).

**Theorem 5.7.** (i) Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be continuous and \( \theta \)-generalized closed. Then, for a \( \theta \)-g-closed set \( A \) of \( X \), \( f(A) \) is \( \theta \)-g-closed in \( Y \).
(ii) Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be strongly \( \theta \)-continuous and closed. Then, \( f \) is \( \theta \)-irresolute.

**Proof.** (i) Left to the reader.

(ii) Let \( B \) be a \( \theta \)-g-closed set of \( (Y, \sigma) \) and let \( U \in \tau \) such that \( f^{-1}(B) \subseteq U \). Put \( H = \text{Cl}_\theta(f^{-1}(B)) \cap (X \setminus U) \). A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is strongly \( \theta \)-continuous if and only if \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( (y, \text{id}) \)-continuous in the sense of Ogata [22, Def. 4.12], where \( y : \tau \rightarrow \mathcal{P}(X) \) is the closure operation and \( \text{id} : \sigma \rightarrow \mathcal{P}(Y) \) is the identity operation. Using [22, Prop. 4.13(i)] and the fact that \( \text{Cl}_\gamma(E) = \text{Cl}_\varphi(E) \) and \( \text{Cl}_{\text{id}}(E) = \text{Cl}(E) \) for the closure operation \( \gamma \), the identity operation \( \text{id} \) and the subset \( E \), we get \( f(H) \subseteq f(\text{Cl}_\theta(f^{-1}(B))) \cap f(X \setminus B) \subseteq \text{Cl}(f(f^{-1}(B))) \cap (X \setminus B) \subseteq \text{Cl}(B) \setminus B \subseteq \text{Cl}_\theta(B) \setminus B \). By Lemma 4.2, \( f(H) = \emptyset \) since \( f(H) \) is closed. We have \( H = \emptyset \) and hence \( \text{Cl}_\varphi(f^{-1}(B)) \subseteq U \). Therefore, \( f^{-1}(B) \subseteq \text{TGC}(X, \tau) \).

**Corollary 5.8.** (i) Under the same assumptions of Theorem 5.6, if \( (X, \tau) \) is \( T_{1/2} \), then \( (Y, \sigma) \) is \( T_{1/2} \).

(ii) Under the same assumptions of Theorem 5.7(ii), if \( (X, \tau) \) is \( T_{1/2} \) and \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \theta \)-g-open, then \( (Y, \sigma) \) is \( T_{1/2} \).

**Proposition 5.9.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( \theta \)-generalized continuous function and let \( H \) be a \( \theta \)-closed subset of \( X \). Then the restriction \( f \mid H : (H, \tau \mid H) \rightarrow (Y, \sigma) \) is \( \theta \)-generalized continuous.

**Proof.** Let \( F \) be a closed subset of \( (Y, \sigma) \). By Proposition 3.11, \( H_1 = f^{-1}(F) \cap H \) is \( \theta \)-g-closed in \( (X, \tau) \). Then, by Theorem 3.13(ii), \( H_1 \) is \( \theta \)-g-closed in \( (H, \tau \mid H) \). Since \( (f \mid H)^{-1}(F) = H_1 \), \( f \mid H \) is \( \theta \)-g-continuous.

Next, we offer the following “Pasting Lemma” for \( \theta \)-g-continuous functions.

**Proposition 5.10.** Let \( (X, \tau) \) be a topological space such that \( X = A \cup B \), where both \( A, B \in \text{TGC}(X) \) and \( A, B \in \tau \). Let \( f : (A, \tau \mid A) \rightarrow (Y, \sigma) \) and \( g : (B, \tau \mid B) \rightarrow (Y, \sigma) \) be \( \theta \)-g-continuous functions such that \( f(x) = g(x) \) for every \( x \in A \cap B \). Then the combination \( \alpha : (X, \tau) \rightarrow (Y, \sigma) \) is \( \theta \)-g-continuous, where \( \alpha(x) = f(x) \) for any \( x \in A \) and \( \alpha(y) = g(y) \) for any \( y \in B \).

**Definition 8.** A subset \( A \) of \( (X, \tau) \) is called \( \theta \)-g-open (= \( \theta \)-g-open) if its complement \( X \setminus A \) is \( \theta \)-g-closed in \( (X, \tau) \).

**Theorem 5.11.** (i) A subset \( A \) of \( (X, \tau) \) is \( \theta \)-g-open if and only if \( F \subseteq \text{Int}_\varphi(A) \), whenever \( F \subseteq A \) and \( F \) is closed in \( (X, \tau) \).

(ii) If \( A \) is \( \theta \)-g-open in \( (X, \tau) \) and \( B \) is \( \theta \)-g-open in \( (Y, \sigma) \), then \( A \times B \) is \( \theta \)-g-open in the product space \( (X \times Y, \tau \times \sigma) \).

**Proof.** (i) Obvious.

(ii) Let \( F \) be a closed subset of \( (X \times Y, \tau \times \sigma) \) such that \( F \subseteq A \times B \). For each \( (x, y) \in F \), \( \text{Cl}(|\{x\}| \times \text{Cl}(|\{y\}|) \subseteq \text{Cl}(F) = F \subseteq A \times B \). Then the two closed sets \( \text{Cl}(|\{x\}|) \) and \( \text{Cl}(|\{y\}|) \) are contained in \( A \) and \( B \), respectively. By assumption, \( \text{Cl}(|\{x\}|) \subseteq \text{Int}_\varphi(A) \) and \( \text{Cl}(|\{y\}|) \subseteq \text{Int}_\varphi(B) \) hold. This implies that, for each \( (x, y) \in F \), \( (x, y) \in \text{Int}_\varphi(A) \times \text{Int}_\varphi(B) \subseteq \text{Int}_\varphi(A \times B) \) and hence \( F \subseteq \text{Int}_\varphi(A \times B) \). By (i) it is clear that \( A \times B \) is \( \theta \)-g-open.
**Proposition 5.12.** The projection \( p : (X \times Y, \tau \times \sigma) \to (X, \tau) \) is a \( \theta\)-g-irresolute map.

**Proof.** By definition and Theorem 5.11(ii), for a \( \theta\)-generalized closed set \( F \) of \((X, \tau)\), \( p^{-1}(x \setminus F) = (X \setminus F) \times Y \) is \( \theta\)-g-open in \((X \times Y, \tau \times \sigma)\). Therefore, \( p^{-1}(F) = F \times Y = X \times Y \setminus (p^{-1}(X \setminus F)) \) is \( \theta\)-generalized closed. \(\square\)


**Definition 9.** (cf. [15]). A topological space \( X \) is called TGO-connected (respectively, GO-connected [15]) if \( X \) cannot be written as a disjoint union of two nonempty \( \theta\)-g-open (respectively, \( g\)-open) sets. A subset of \( X \) is called TGO-connected if it is connected as a subspace.

Clearly, every TGO-connected space is connected. The space in [3, Ex. 11] shows that there are connected spaces which are not TGO-connected. Since every \( \theta\)-generalized closed set is \( g\)-closed, every GO-connected space is TGO-connected. Thus, we have the following implications and none of them is reversible.

\[
\text{GO-connected} \implies \text{TGO-connected} \implies \text{Connected}
\]

**Example 6.1.** Let \( X = \{a, b, c, d\} \) and let \( \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\} \). Since \( \{c\} \) is both \( g\)-closed and \( g\)-open, \( X \) is not GO-connected. Note that TGC\((X) = \{\emptyset, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\} \). Hence, \( X \) is TGO-connected.

**Observation 6.2.** (i) [3, Prop. 10]. For a topological space \((X, \tau)\), the following conditions are equivalent.
\begin{enumerate}
  \item \( X \) is TGO-connected;
  \item the only subsets of \( X \), which are both \( \theta\)-g-open and \( \theta\)-g-closed, are \( \emptyset \) and \( X \);
  \item each \( \theta\)-generalized continuous function of \( X \) into a discrete space \( Y \), with at least two points, is constant.
\end{enumerate}

(ii) [3, Prop. 12]. If \((X, \tau)\) is a \( T_{1/2} \)-space, then the following conditions are equivalent
\begin{enumerate}
  \item \( X \) is GO-connected;
  \item \( X \) is TGO-connected;
  \item \( X \) is connected.
\end{enumerate}

(iii) A regular space \( X \) is GO-connected if and only if \( X \) is TGO-connected.

(iv) Let \( f : (X, \tau) \to (Y, \sigma) \) be a surjection. Then
\begin{enumerate}
  \item If \( f \) is \( \theta\)-generalized continuous and \( X \) is TGO-connected, then \( Y \) is connected.
  \item If \( f \) is \( \theta\)-g-irresolute and \( X \) is TGO-connected, then \( Y \) is TGO-connected.
\end{enumerate}

**Corollary 6.3.** If the product space \((X \times Y, \tau \times \sigma)\) is TGO-connected, then its factor space \((X, \tau)\) is TGO-connected.

**Theorem 6.4.** Let \( f : (X, \tau) \to (Y, \sigma) \) be \( \theta\)-g-continuous. Then the image of every \( \theta\)-closed, TGO-connected subset of \((X, \tau)\) is connected in \((Y, \sigma)\).
Proof. Let $H$ be a $\theta$-closed and TGO-connected set in $(X, \tau)$. Then, by Proposition 5.9, the restriction of $f$ to $H$, $f \mid H : (H, \tau \mid H) \to (Y, \sigma)$, is $\theta$-g-continuous. For $f$, a function $r_H(f) : (H, \tau \mid H) \to (f(H), \sigma \mid f(H))$ is well defined by $(r_H(f))(x) = f(x)$ for any $x \in H$. Since $f \mid H = j \circ r_H(f)$, where $j : (f(H), \tau \mid f(H)) \to (Y, \sigma)$ is an inclusion. Then it is clear that $r_H(f)$ is $\theta$-g-continuous. In fact, for an open set $V$ of $(f(H), \sigma \mid f(H))$, take an open set $G \in \tau$ such that $G \cap f(H) = V$. Then $r_H(f)^{-1}(V) = (f \mid H)^{-1}(G)$ is $\theta$-g-open. Now, by Observation 6.2(iv), $(f(H), \sigma \mid f(H))$ is connected and hence $f(H)$ is a connected subset of $(Y, \sigma)$. □

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