EXISTENCE AND UNIQUENESS THEOREM FOR A SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS

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Abstract. By using the method of successive approximation, we prove the existence and uniqueness of a solution of the fuzzy differential equation \( x'(t) = f(t, x(t)), \ x(t_0) = x_0 \). We also consider an \( \epsilon \)-approximate solution of the above fuzzy differential equation.

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1. Introduction. The differential equation

\[
    x'(t) = f(t, x(t)), \quad x(t_0) = x_0
\]

has a solution provided \( f \) is continuous and satisfies a Lipschitz condition by C. Corduneanu [2]. The definition given here generalizes that of Aumann [1] for set-valued mappings. Kaleva [3] discussed the properties of differentiable fuzzy set-valued mappings and gave the existence and uniqueness theorem for a solution of the fuzzy differential equation \( x'(t) = f(t, x(t)) \) when \( f \) satisfies the Lipschitz condition. Also, in [4], he dealt with fuzzy differential equations on locally compact spaces. Park [6, 7] showed existence of solutions for fuzzy integral equations and a fixed point theorem for a pair of generalized nonexpansive fuzzy mappings.

In this paper, we prove the existence and uniqueness theorem of a solution to the fuzzy differential equation (1.1), where \( f : I \times E^n \rightarrow E^n \) is levelwise continuous and satisfies a generalized Lipschitz condition.

Under some hypotheses, we consider an \( \epsilon \)-approximate solution of the above fuzzy differential equation.

2. Preliminaries. Let \( P_k(R^n) \) denote the family of all nonempty compact convex subsets of \( R^n \) and define the addition and scalar multiplication in \( P_k(R^n) \) as usual. Let \( A \) and \( B \) be two nonempty bounded subsets of \( R^n \). The distance between \( A \) and \( B \) is defined by the Hausdorff metric

\[
    d(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\},
\]

where \( \|\cdot\| \) denotes the usual Euclidean norm in \( R^n \). Then it is clear that \( (P_k(R^n), d) \) becomes a metric space.

**Theorem 2.1** [8]. The metric space \( (P_k(R^n), d) \) is complete and separable.
Let $T = [c, d] \subset \mathbb{R}$ be a compact interval and denote

$$E^n = \{ u : \mathbb{R}^n \rightarrow [0, 1] \mid u \text{ satisfies (i)-(iv) below} \}, \quad (2.2)$$

where

- (i) $u$ is normal, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,
- (ii) $u$ is fuzzy convex,
- (iii) $u$ is upper semicontinuous,
- (iv) $[u]^0 = \text{cl}\{x \in \mathbb{R}^n \mid u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{ x \in \mathbb{R}^n \mid u(x) \geq \alpha \}$, then from (i)-(iv), it follows that the $\alpha$-level set $[u]^\alpha \in \mathcal{P}_k(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

If $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a function, then, according to Zadeh's extension principle, we can extend $g$ to $E^n \times E^n \rightarrow E^n$ by the equation

$$g(u, v)(z) = \sup_{z = g(x, y)} \min\{u(x), v(y)\}. \quad (2.3)$$

It is well known that

$$[g(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha) \quad (2.4)$$

for all $u, v \in E^n$, $0 \leq \alpha \leq 1$ and $g$ is continuous. Especially for addition and scalar multiplication, we have

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha, \quad (2.5)$$

where $u, v \in E^n$, $k \in \mathbb{R}$, $0 \leq \alpha \leq 1$.

**Theorem 2.2** [5]. If $u \in E^n$, then

1. $[u]^\alpha \in \mathcal{P}_k(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$,
2. $[u]^\alpha \subset [u]^\alpha_1$ for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$,
3. if $\{\alpha_k\} \subset [0, 1]$ is a nondecreasing sequence converging to $\alpha > 0$, then

$$[u]^\alpha = \bigcap_{k \geq 1} [u]^\alpha_k. \quad (2.6)$$

Conversely, if $\{A^\alpha \mid 0 \leq \alpha \leq 1\}$ is a family of subsets of $\mathbb{R}^n$ satisfying (1)-(3), then there exists $u \in E^n$ such that

$$[u]^\alpha = A^\alpha \quad \text{for } 0 < \alpha \leq 1 \quad (2.7)$$

and

$$[u]^0 = \bigcup_{0 < \alpha \leq 1} A^\alpha \subset A^0. \quad (2.8)$$

Define $D : E^n \times E^n \rightarrow \mathbb{R}^+ \cup \{0\}$ by the equation

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha), \quad (2.9)$$

where $d$ is the Hausdorff metric defined in $\mathcal{P}_k(\mathbb{R}^n)$.

The following definitions and theorems are given in [3].

**Definition 2.1.** A mapping $F : T \rightarrow E^n$ is strongly measurable if, for all $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha : T \rightarrow \mathcal{P}_k(\mathbb{R}^n)$ defined by

$$F_\alpha(t) = [F(t)]^\alpha \quad (2.10)$$
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is Lebesgue measurable, when $P_k(R^n)$ is endowed with the topology generated by the
Hausdorff metric $d$.

**Definition 2.2.** A mapping $F : T \to E^n$ is called *levelwise continuous* at $t_0 \in T$ if
the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is continuous at $t = t_0$ with respect to the
Hausdorff metric $d$ for all $\alpha \in [0, 1]$.

A mapping $F : T \to E^n$ is called *integrably bounded* if there exists an integrable
function $h$ such that $\|x\| \leq h(t)$ for all $x \in F_0(t)$.

**Definition 2.3.** Let $F : T \to E^n$. The integral of $F$ over $T$, denoted by $\int_T F(t)$ or
$\int_0^T F(t) dt$, is defined levelwise by the equation

$$\left( \int_T F(t) dt \right)^\alpha = \int_T F_\alpha(t) dt$$

$$(2.11)$$

for all $0 < \alpha \leq 1$.

A strongly measurable and integrably bounded mapping $F : T \to E^n$ is said to be
integrable over $T$ if $\int_T F(t) dt \in E^n$.

**Theorem 2.3.** If $F : T \to E^n$ is strongly measurable and integrably bounded, then $F$
is integrable.

It is known that $[\int_T F(t) dt]^0 = \int_T F_0(t) dt$.

**Theorem 2.4.** Let $F, G : T \to E^n$ be integrable, and $\lambda \in R$. Then
(i) $\int_T (F(t) + G(t)) dt = \int_T F(t) dt + \int_T G(t) dt$.
(ii) $\int_T \lambda F(t) dt = \lambda \int_T F(t) dt$.
(iii) $D(F, G)$ is integrable.
(iv) $D(\int_T F(t) dt, \int_T G(t) dt) \leq \int_T D(F, G)(t) dt$.

**Definition 2.4.** A mapping $F : T \to E^n$ is called *differentiable* at $t_0 \in T$ if, for
any $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at
point $t_0$ with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) \mid \alpha \in [0, 1]\}$ define a fuzzy number
$F(t_0) \in E^n$.

If $F : T \to E^n$ is differentiable at $t_0 \in T$, then we say that $F'(t_0)$ is the *fuzzy derivative*
of $F(t)$ at the point $t_0$.

**Theorem 2.5.** Let $F : T \to E^n$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$. Then
$f_\alpha$ and $g_\alpha$ are differentiable and $[F'(t)]^\alpha = [f_\alpha'(t), g_\alpha'(t)]$.

**Theorem 2.6.** Let $F : T \to E^n$ be differentiable and assume that the derivative $F'$ is
integrable over $T$. Then, for each $s \in T$, we have

$$F(s) = F(a) + \int_a^s F'(t) dt.$$  

$$(2.12)$$

**Definition 2.5.** A mapping $f : T \times E^n \to E^n$ is called *levelwise continuous* at point
$(t_0, x_0) \in T \times E^n$ provided, for any fixed $\alpha \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists a
\[ \delta(\epsilon, \alpha) > 0 \text{ such that} \]
\[ d\left( [f(t,x)]^\alpha, [f(t_0,x_0)]^\alpha \right) < \epsilon \] (2.13)
whenever \(|t - t_0| < \delta(\epsilon, \alpha)\) and \(d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha)\) for all \(t \in T, x \in E^n\).

3. **Fuzzy differential equations.** Assume that \(f : I \times E^n \to E^n\) is levelwise continuous, where the interval \(I = \{t : |t - t_0| \leq \delta \leq a\}\). Consider the fuzzy differential equation (1.1) where \(x_0 \in E^n\). We denote \(J_0 = I \times B(x_0, b)\), where \(a > 0, b > 0, x_0 \in E^n\),
\[ B(x_0, b) = \{x \in E^n | D(x, x_0) \leq b\}. \] (3.1)

**Definition 3.1.** A mapping \(x : I \to E^n\) is a solution to the problem (1.1) if it is levelwise continuous and satisfies the integral equation
\[ x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \quad \text{for all} \ t \in I. \] (3.2)

According to the method of successive approximation, let us consider the sequence \(\{x_n(t)\}\) such that
\[ x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))ds, \quad n = 1, 2, \ldots, \] (3.3)
where \(x_0(t) \equiv x_0, \ t \in I\).

**Theorem 3.1.** Assume that
(i) a mapping \(f : J_0 \to E^n\) is levelwise continuous,
(ii) for any pair \((t, x), (t, y) \in J_0\), we have
\[ d\left( [f(t, x)]^\alpha, [f(t, y)]^\alpha \right) \leq Ld([x]^\alpha, [y]^\alpha), \] (3.4)
where \(L > 0\) is a given constant and for any \(\alpha \in [0, 1]\).

Then there exists a unique solution \(x = x(t)\) of (1.1) defined on the interval
\[ |t - t_0| \leq \delta = \min \left\{ a, \frac{b}{M} \right\}, \] (3.5)
where \(M = D(f(t,x), \hat{o}), \hat{o} \in E^n\) such that \(\hat{o}(t) = 1\) for \(t = 0\) and 0 otherwise and for any \((t, x) \in J_0\).

Moreover, there exists a fuzzy set-valued mapping \(x : I \to E^n\) such that \(D(x_n(t), x(t)) \to 0\ on |t - t_0| \leq \delta\ as n \to \infty\).

**Proof.** Let \(t \in I\), from (3.3), it follows that, for \(n = 1\),
\[ x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0)ds \] (3.6)
which proves that \(x(t)\) is levelwise continuous on \(|t - t_0| \leq a\) and, hence on \(|t - t_0| \leq \delta\).
Moreover, for any \(\alpha \in [0, 1]\), we have
\[ d([x_1(t)]^\alpha, [x_0]^\alpha) = d\left( \left[ \int_{t_0}^{t} f(s, x_0)ds \right]^\alpha, 0 \right) \leq \int_{t_0}^{t} d\left( [f(s, x_0)]^\alpha, 0 \right)ds \] (3.7)
and by the definition of \(D\), we get
\[ D(x_1(t), x_0) \leq M|t - t_0| \leq M\delta = b \] (3.8)
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if \(|t - t_0| \leq \delta\), where \(M = D(f(t,x), \hat{o})\), \(\hat{o} \in E^n\) and for any \((t,x) \in J_0\).

Now, assume that \(x_{n-1}(t)\) is levelwise continuous on \(|t - t_0| \leq \delta\) and that
\[
D(x_{n-1}(t), x_0) \leq M|t - t_0| \leq M\delta = b
\] (3.9)

if \(|t - t_0| \leq \delta\), where \(M = D(f(t,x), \hat{o})\), \(\hat{o} \in E^n\) and for any \((t,x) \in J_0\).

From (3.3), we deduce that \(x_n(t)\) is levelwise continuous on \(|t - t_0| \leq \delta\) and that
\[
D(x_n(t), x_0) \leq M|t - t_0| \leq M\delta = b
\] (3.10)

if \(|t - t_0| \leq \delta\), where \(M = D(f(t,x), \hat{o})\), \(\hat{o} \in E^n\) and for any \((t,y) \in J_0\).

Consequently, we conclude that \(\{x_n(t)\}\) consists of levelwise continuous mappings on \(|t - t_0| \leq \delta\) and that
\[
(t, x_n(t)) \in J_0, \quad |t - t_0| \leq \delta, \quad n = 1, 2, \ldots
\] (3.11)

Let us prove that there exists a fuzzy set-valued mapping \(x : I \to E^n\) such that \(D(x_n(t), x(t)) \to 0\) uniformly on \(|t - t_0| \leq \delta\) as \(n \to \infty\). For \(n = 2\), from (3.3),
\[
x_2(t) = x_0 + \int_{t_0}^t f(s, x_1(s)) ds.
\] (3.12)

From (3.6) and (3.12), we have
\[
d\left([x_2(t)]^\alpha, [x_1(t)]^\alpha\right) = d\left([\int_{t_0}^t f(s, x_1(s)) ds]^\alpha, [\int_{t_0}^t f(s, x_0(s)) ds]^\alpha\right)
\leq \int_{t_0}^t d\left([f(s, x_1(s))]^\alpha, [f(s, x_0(s))]^\alpha\right) ds
\] (3.13)

for any \(\alpha \in [0,1]\).

According to the condition (3.4), we obtain
\[
d\left([x_2(t)]^\alpha, [x_1(t)]^\alpha\right) \leq \int_{t_0}^t Ld\left([x_1(s)]^\alpha, [x_0(s)]^\alpha\right) ds
\] (3.14)

and by the definition of \(D\), we obtain
\[
D(x_2(t), x_1(t)) \leq L \int_{t_0}^t D(x_1(s), x_0(s)) ds.
\] (3.15)

Now, we can apply the first inequality (3.8) in the right-hand side of (3.15) to get
\[
D(x_2(t), x_1(t)) \leq ML \frac{|t - t_0|^2}{2!} \leq ML \frac{\delta^2}{2!}.
\] (3.16)

Starting from (3.8) and (3.16), assume that
\[
D(x_n(t), x_{n-1}(t)) \leq ML^{n-1} \frac{|t - t_0|^n}{n!} \leq ML^{n-1} \frac{\delta^n}{n!}
\] (3.17)

and let us prove that such an inequality holds for \(D(x_{n+1}(t), x_n(t))\).
Indeed, from (3.3) and condition (3.4), it follows that
\[
d\left(\left[x_{n+1}(t)\right]^\alpha,\left[x_n(t)\right]^\alpha\right) = d\left(\left[\int_{t_0}^{t} f(s,x_n(s))ds\right]^\alpha,\left[\int_{t_0}^{t} f(s,x_{n-1}(s))ds\right]^\alpha\right)
\leq \int_{t_0}^{t} d\left(\left[f(s,x_n(s))\right]^\alpha,\left[f(s,x_{n-1}(s))\right]^\alpha\right)ds
\leq \int_{t_0}^{t} Ld\left(\left[x_n(s)\right]^\alpha,\left[x_{n-1}(s)\right]^\alpha\right)ds
\] (3.18)
for any \(\alpha \in [0,1]\) and from the definition of \(D\), we have
\[
D\left(x_{n+1}(t),x_n(t)\right) \leq L \int_{t_0}^{t} D\left(\left[x_n(s)\right]^\alpha,\left[x_{n-1}(s)\right]^\alpha\right)ds.
\] (3.19)

According to (3.17), we get
\[
D\left(x_{n+1}(t),x_n(t)\right) \leq ML^n \int_{t_0}^{t} |s-t_0|^n ds = \frac{ML^n |t-t_0|^{n+1}}{(n+1)!} \leq \frac{ML^n \delta^{n+1}}{(n+1)!}.
\] (3.20)

Consequently, inequality (3.17) holds for \(n = 1,2,\ldots\). We can also write
\[
D\left(x_n(t),x_{n-1}(t)\right) \leq \frac{M (L\delta)^n}{n!}
\] (3.21)
for \(n = 1,2,\ldots\), and \(|t-t_0| \leq \delta\).

Let us mention now that
\[
x_n(t) = x_0 + [x_1(t) - x_0] + \cdots + [x_n(t) - x_{n-1}(t)],
\] (3.22)
which implies that the sequence \(\{x_n(t)\}\) and the series
\[
x_0 + \sum_{n=1}^{\infty} [x_n(t) - x_{n-1}(t)]
\] (3.23)
have the same convergence properties.

From (3.21), according to the convergence criterion of Weierstrass, it follows that the series having the general term \(x_n(t) - x_{n-1}(t)\), so \(D(x_n(t),x_{n-1}(t)) \to 0\) uniformly on \(|t-t_0| \leq \delta\) as \(n \to \infty\).

Hence, there exists a fuzzy set-valued mapping \(x : I \to E^n\) such that \(D(x_n(t),x(t)) \to 0\) uniformly on \(|t-t_0| \leq \delta\) as \(n \to \infty\).

From (3.4), we get
\[
d\left(\left[f(t,x_n(t))\right]^\alpha,\left[f(t,x(t))\right]^\alpha\right) \leq Ld\left(\left[x_n(t)\right]^\alpha,\left[x(t)\right]^\alpha\right)
\] (3.24)
for any \(\alpha \in [0,1]\). By the definition of \(D\),
\[
D\left(f(t,x_n(t)),f(t,x(t))\right) \leq LD(x_n(t),x(t)) \to 0
\] (3.25)
uniformly on \(|t-t_0| \leq \delta\) as \(n \to \infty\).

Taking (3.25) into account, from (3.3), we obtain, for \(n \to \infty\),
\[
x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds.
\] (3.26)
Consequently, there is at least one levelwise continuous solution of (1.1).

We want to prove now that this solution is unique, that is, from

\[ y(t) = x_0 + \int_{t_0}^{t} f(s, y(s))\, ds \]  

(3.27)
on \mid t - t_0 \mid \leq \delta$, it follows that $D(x(t), y(t)) = 0$. Indeed, from (3.3) and (3.27), we obtain

\[ d\left(\left[ y(t)\right]^\alpha, \left[ x_n(t)\right]^\alpha\right) = d\left(\left[ \int_{t_0}^{t} f(s, y(s))\, ds\right]^\alpha, \left[ \int_{t_0}^{t} f(s, x_{n-1}(s))\, ds\right]^\alpha\right) \]

\[ \leq \int_{t_0}^{t} d\left(\left[ f(s, y(s))\right]^\alpha, \left[ f(s, x_{n-1}(s))\right]^\alpha\right)\, ds \]

\[ \leq \int_{t_0}^{t} Ld\left(\left[ y(s)\right]^\alpha, \left[ x_{n-1}(s)\right]^\alpha\right)\, ds \]

(3.28)

for any $\alpha \in [0, 1], n = 1, 2, \ldots$

By the definition of $D$, we obtain

\[ D\left( y(t), x_1(t) \right) \leq bL|t - t_0| \]  

(3.30)
on \mid t - t_0 \mid \leq \delta$. Now, assume that

\[ D\left( y(t), x_n(t) \right) \leq bL^n \frac{|t - t_0|^n}{n!} \]  

(3.31)
on the interval \mid t - t_0 \mid \leq \delta. From

\[ D\left( y(t), x_{n+1}(t) \right) \leq L \int_{t_0}^{t} D\left( y(s), x_n(s) \right)\, ds \]  

(3.32)

and (3.31), one obtains

\[ D\left( y(t), x_{n+1}(t) \right) \leq bL^{n+1} \frac{|t - t_0|^{n+1}}{(n+1)!}. \]  

(3.33)

Consequently, (3.31) holds for any $n$, which leads to the conclusion

\[ D\left( y(t), x_n(t) \right) = D(x(t), x_n(t)) \rightarrow 0 \]  

(3.34)
on the interval \mid t - t_0 \mid \leq \delta as $n \rightarrow \infty$.

This proves the uniqueness of the solution for (1.1).

**Definition 3.2.** A mapping $x : L \to E^n$ is an $\epsilon$-approximate solution of (1.1) if the following properties hold

(a) $x(t)$ is levelwise continuous on \mid t - t_0 \mid \leq \delta,
(b) the derivative $x'(t)$ exists and it is levelwise continuous,
(c) for all $t$ for which $x'(t)$ is defined, we have

\[ D\left( x'(t), f(t, x(t)) \right) < \epsilon. \]  

(3.35)
**Theorem 3.2.** A mapping \( f : J_0 \to \mathbb{E}^n \) is levelwise continuous, and let \( \epsilon > 0 \) be arbitrary. Then there exists at least one \( \epsilon \)-approximate solution of (1.1), defined on \( |t - t_0| \leq \delta = \min\{a, b/M\} \), where \( M = D(f(t,x),\hat{o}) \), \( \hat{o} \in \mathbb{E}^n \) and for any \( (t,x) \in J_0 \).

**Proof.** Inasmuch as a mapping \( f : J_0 \to \mathbb{E}^n \) is a levelwise continuous on a compact set \( J_0 \), it follows that \( f(t,x) \) is uniformly levelwise continuous. Consequently, for any \( \alpha \in [0,1] \), we can find \( \delta > 0 \) such that \( d([f(t,x)]^\alpha,[f(s,y)]^\alpha) < \epsilon \).

Now, we construct the approximate solution for \( t \in [t_0,t_0 + \delta] \), the construction being completely similar for \( t \in [t_0 - \delta,t_0] \).

Let us consider a division

\[
 t_0 < t_1 < \cdots < t_n = t_0 + \delta
\]

of \( [t_0,t_0 + \delta] \) such that

\[
 \max_k (t_k - t_{k-1}) < \lambda = \min \left\{ \delta, \frac{\delta}{M} \right\}.
\]

We define a mapping \( x : I \to \mathbb{E}^n \) as follows

\[
 x(t_0) = x_0,
\]
\[
 x(t) = x(t_k) + f(t_k,x(t_k))(t - t_k)
\]

on \( t_k < t \leq t_{k+1}, k = 0,1,\ldots,n-1. \)

It is obvious that a mapping \( x : I \to \mathbb{E}^n \) satisfies the first two properties from the definition of an \( \epsilon \)-approximate solution.

Now, we want to prove that the last property is also fulfilled. Indeed, \( x'(t) = f(t_k, x(t_k)) \) on \( (t_k,t_{k+1}) \) and for any \( \alpha \in [0,1] \),

\[
 d\left([x'(t)]^\alpha,\left[f(t,x(t))\right]^\alpha\right) = d\left([f(t_k,x(t_k))]^\alpha,\left[f(t,x(t))\right]^\alpha\right) < \epsilon
\]

since \( |t - t_k| < \lambda \leq \delta \),

\[
 d\left([x(t)]^\alpha,[x(t_k)]^\alpha\right) \leq d\left([f(t_k,x(t_k))]^\alpha,0\right)|t - t_k| < M\lambda \leq \delta.
\]

Thus, by the definition of \( D \), we have

\[
 D(x'(t),f(t,x(t))) < \epsilon
\]

on \( |t - t_0| < \delta \) and \( (t,x) \in J_0 \).

Theorem 3.2 is completely proved. \( \square \)

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