RECURSIVE FORMULAE FOR THE MULTIPLICATIVE PARTITION FUNCTION

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(Received 28 February 1996)

Abstract. For a positive integer \( n \), let \( f(n) \) be the number of essentially different ways of writing \( n \) as a product of factors greater than 1, where two factorizations of a positive integer are said to be essentially the same if they differ only in the order of the factors. This paper gives a recursive formula for the multiplicative partition function \( f(n) \).

Keywords and phrases. Partitions, multiplicative partitions.

1991 Mathematics Subject Classification. 11P82.

A multi-partite number of order \( j \) is a \( j \)-dimensional vector, the components of which are nonnegative integers. A partition of \((n_1, n_2, \ldots, n_j)\) is a solution of the vector equation

\[
\sum_k (n_{1k}, n_{2k}, \ldots, n_{jk}) = (n_1, n_2, \ldots, n_j)
\]

in multi-partition numbers other than \((0,0,\ldots,0)\). Two partitions which differ only in the order of the multi-partite numbers are regarded as identical. We denote by \( p(n_1, n_2, \ldots, n_j) \) the number of different partitions of \((n_1, n_2, \ldots, n_j)\). For example, \( p(3) = 3 \) since \( 3 = 3 \) and \( p(2,1) = 4 \) since \( 2 = (2,0) + (0,1) = (1,0) + (1,0) + (1,0) = (1,0) + (1,1) \). Let \( f(1) = 1 \) and for any integer \( n > 1 \), let \( f(n) \) be the number of essentially different ways of writing \( n \) as a product of factors greater than 1, where two factorizations of a positive integer are said to be essentially the same if they differ only in the order of the factors. For example, \( f(12) p(2,1) = 4 \) since \( 12 = 2 \cdot 6 = 3 \cdot 4 = 2 \cdot 2 \cdot 3 \). In general, if \( n = p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j} \), then \( f(n) = p(n_1, n_2, \ldots, n_j) \).

We find recursive formulas for the multi-partite partition function \( p(n_1, n_2, \ldots, n_j) \). The most useful formula known to this day for actual evaluation of the multi-partite partition function is presented in Theorem 4.

For convenience, we define some sets used in this paper. For a positive integer \( r \), let \( M^0_0 \) be the set of \( r \)-dimensional vectors with nonnegative integer components and \( M_r \) be the set of \( r \)-dimensional vectors with nonnegative integer components not all of which are zero. The following three theorems are well known.

**Theorem 1** (Euler [3]; see also [1, p. 2]). If \( n \geq 0 \), then

\[
p(n) = \sum_{m=1}^{\infty} (-1)^{m+1} \left( p(n - \frac{1}{2} m (3m-1)) + p(n - \frac{1}{2} (3m+1)) \right),
\]

where we recall that \( p(k) = 0 \) for all negative integers \( k \).
THEOREM 2. If \( n \geq 0 \), then \( p(0) = 1 \) and
\[
n \cdot p(n) = \sum_{k=1}^{n} \sigma(k) \cdot p(n-k),
\]
where \( \sigma(m) = \sum_{d|m} d \).

THEOREM 3 ([1, Ch. 12]). If \( g(x_1, x_2, \ldots, x_r) \) is the generating function for \( p(\vec{n}) \) and \( |x_i| < 1 \) for \( i \leq r \), then
\[
g(x_1, x_2, \ldots, x_r) = \prod_{\vec{n} \in M_r} \frac{1}{1 - x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}} = 1 + \sum_{\vec{m} \in M_r} p(\vec{m}) x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}.
\]

Similarly, we can extend the equation of Theorem 2 to multi-partite numbers as follows.

THEOREM 4. For \( \vec{n} \in M_r \), we have
\[
n_i \cdot p(\vec{n}) = \sum_{\vec{l} \in M_r} \frac{\sigma(\gcd(\vec{l}))}{\gcd(\vec{l})} \cdot l_i \cdot p(\vec{n} - \vec{l}).
\]

PROOF. Let \( g(x_1, x_2, \ldots, x_r) \) be the function defined in Theorem 3. Taking the \( i \)th partial logarithmic derivative of the product formula for \( g(x_1, x_2, \ldots, x_r) \) in (4), we get
\[
\frac{\partial g(x_1, x_2, \ldots, x_r)}{\partial x_i} \cdot \frac{x_i}{g(x_1, x_2, \ldots, x_r)} = \sum_{\vec{l} \in M_r} \frac{l_i \cdot \prod_{j=1}^{r} x_j^{l_j}}{1 - \prod_{j=1}^{r} x_j^{l_j}} = \sum_{\vec{l} \in M_r, k=1}^{\infty} l_i \cdot \left( \prod_{j=1}^{r} x_j^{l_j} \right)^k.
\]

Taking the \( i \)th partial derivative of the right-hand side of (4), we get
\[
\sum_{\vec{n} \in M_r} n_i \cdot p(\vec{n}) x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} = \frac{\partial g(x_1, x_2, \ldots, x_r)}{\partial x_i} \cdot x_i
\]
\[
= g(x_1, x_2, \ldots, x_r) \sum_{\vec{l} \in M_r} \sum_{k=1}^{\infty} t_i \cdot \left( \prod_{j=1}^{r} x_j^{l_j} \right)^k = \left( \sum_{\vec{m} \in M_r^0} p(\vec{m}) x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} \right) \sum_{\vec{l} \in M_r} t_i \cdot \left( \prod_{j=1}^{r} x_j^{l_j} \right)^k.
\]

Comparing the coefficients of both sides of (7), we get
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\[
n_i \cdot p(\vec{n}) = \sum_{\vec{m}, \vec{t} \in M_0, k \in M_1} t_i \cdot p(\vec{m})
\]
\[
= \sum_{\vec{m} + k\vec{t} = \vec{n}} p(\vec{n} - \vec{t}) \sum_{k \mid \gcd(\vec{t})} l_i \cdot \frac{\sigma(\gcd(\vec{t}))}{\gcd(\vec{t})} \cdot l_i \cdot p(\vec{n} - \vec{t}).
\]

(8)

The theorem is proved.

\[\square\]

**Corollary 5.** For \(\vec{n} \in M_r\), we have

\[
\left(\sum_{i=1}^r n_i\right) \cdot p(\vec{n}) = \sum_{l_j \leq n_j \text{ for } j \leq r \text{ and } l_j \in M_r} \sigma(\gcd(\vec{l})) \cdot \left(\sum_{i=1}^r l_i \right) \cdot p(\vec{n} - \vec{t}).
\]

(9)

For positive integers \(m\) and \(n\), let

\[
(m, n)_m = \max_{k \mid \text{m}} k.
\]

(10)

The following properties of \((m, n)_m\) are easy to obtain:

1. \((m, p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}) = \gcd(m, n_1, n_2, \ldots, n_k)\)
2. \((m, nk)_m = \gcd((m, n)_m, (mk)_m)\) for \(\gcd(n, k) = 1\)
3. \((mk, n)_m = (m, n)_m \cdot (k, n)_m\) for \(\gcd(m, k) = 1\).

From the point of view of the multiplicative partition function, Theorem 4 can be restated as the following theorem.

**Theorem 6.** let \(n, t\) be positive integers and let \(p\) be a prime number such that \(p \nmid m\). Then we get

\[
t \cdot f(mp^t) = \sum_{i=1, i \mid \text{m}} \sigma((i, l)_m) i \cdot f \left(\frac{m}{l} p^{t-i}\right).
\]

(11)

In [4], MacMahon presents a table of values of \(f(n)\) for those \(n\) which divide one of \(2^{10} \cdot 3^8, 2^{10} \cdot 3 \cdot 5, 2^9 \cdot 3^2 \cdot 5^1, 2^8 \cdot 3^3 \cdot 5^1, 2^6 \cdot 3^2 \cdot 5^2, 2^5 \cdot 3^3 \cdot 5^2\). In [2], Canfield, Erdös, and Pomerance commented that they doubted the correctness of MacMahon’s figures. Specifically,

\[
p(10, 5) = 3804, \quad \text{not 3737,}\]

(12)

\[
p(9, 8) = 13715, \quad \text{not 13748,}\]

(13)

\[
p(10, 8) = 21893, \quad \text{not 21938,}\]

(14)

\[
p(4, 1, 1) = 38, \quad \text{not 28.}\]

(15)

From Theorem 4 we can easily be sure that Canfield, Erdös and Pomerance comment is true.
References


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