CONVEX AND STARLIKE CRITERIA

HERB SILVERMAN

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ABSTRACT. We investigate an expression involving the quotient of the analytic representations of convex and starlike functions. Sufficient conditions are found for functions to be starlike of a positive order and convex.

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1. Introduction. Let \( S \) denote the class of functions \( f \) normalized by \( f(0) = f'(0) - 1 = 0 \) that are analytic and univalent in the unit disk \( \Delta = \{ z : |z| < 1 \} \). A function \( f \) in \( S \) is said to be starlike of order \( \alpha, 0 \leq \alpha < 1 \), and is denoted by \( S^*(\alpha) \) if \( \text{Re}\{z f''(z)/f'(z)\} > \alpha, z \in \Delta \), and is said to be convex and is denoted by \( K \) if \( \text{Re}\{1 + zf''(z)/f'(z)\} > 0, z \in \Delta \). Mocanu [9] studied linear combinations of the representations of convex and starlike functions and defined the class of \( \alpha \)-convex functions. In [8], it was shown that if

\[
\text{Re}\left[\alpha(1 + zf''(z)/f'(z)) + (1 - \alpha)zf'(z)/f(z)\right] > 0 \quad (1.1)
\]

for \( z \in \Delta \), then \( f \) is starlike for \( \alpha \) real and convex for \( \alpha \geq 1 \).

In this note, we investigate the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. In particular, we consider the class \( G_b \) consisting of normalized functions \( f \) defined by

\[
G_b = \left\{ f : \left| \left( \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \right) - 1 \right| < b, z \in \Delta \right\}. \quad (1.2)
\]

We determine sharp values of \( b \) for which \( G_b \subset S^*(\alpha), 1/2 \leq \alpha < 1 \), and also find values of \( b \) for which \( G_b \subset K \). It is known ([7, 10]) that \( K \subset S^*(1/2) \). We show that \( G_1 \subset S^*(1/2) - K \). We also find values of \( b \) for which \( G_b \) is not starlike and not univalent.

We make use of the following lemma obtained by Jack in [4].

**Lemma A.** Suppose \( \omega \) is analytic for \( |z| \leq r \), \( \omega(0) = 0 \) and \( |\omega(z_0)| = \max_{|z|=r} |\omega(z)| \). Then \( z_0 \omega'(z_0) = k \omega(z_0) \), \( k \geq 1 \).

2. Main results

**Theorem 1.** If \( 0 < b \leq 1 \) and \( G_b \) is defined by (1.2), then \( G_b \subset S^*(2/(1 + \sqrt{1 + 8b})) \). The result is sharp for all \( b \).

We prove this theorem in an equivalent form, which we write as
Theorem 1a. Set \( b = (1 - \alpha)/2\alpha^2, 1/2 \leq \alpha < 1 \). Then \( G_b \subset S^*(\alpha) \), with extremal function \( z/(1 - z)^2(1 - \alpha) \).

Proof of Theorem 1a. It is well known that if \( \omega(z) \) is analytic in \( \Delta \) with \( \omega(0) = 0 \), then \( \Re\left(\frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}\right) > \alpha, z \in \Delta \), if and only if \( \omega(z) \) is a Schwarz function, i.e., \( |\omega(z)| < 1 \) for \( z \in \Delta \) with \( \omega(0) = 0 \). Set

\[
p(z) = \frac{zf'(z)}{f(z)} = \frac{1 + (1 - 2\alpha)\omega(z)}{1 - \omega(z)}
\]

Then

\[
1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}
\]

and

\[
\left|\left(\frac{1+zf''(z)/f'(z)}{zf''(z)/f'(z)}\right) - 1\right| = \left|\frac{zp'(z)}{(p(z))^2}\right| = \left|\frac{2(1 - \alpha)z\omega'(z)}{(1 + (1 - 2\alpha)\omega(z))^2}\right|.
\]

If \( f \notin S^*(\alpha) \), then by Lemma A there is a \( z_0 \in \Delta \) for which \( |\omega(z_0)| = 1 \) and \( z_0\omega'(z_0) \geq \omega(z_0) \). It then follows from (2.3) that \( \frac{z_0 \omega'(z_0)}{(p(z))^2} \geq \frac{2(1 - \alpha)}{(2\alpha)^2} \) which contradicts our hypothesis. This completes the proof.

Corollary 1. \( G_1 \subset S^*(1/2) \).

Proof. Set \( b = 1 \) in Theorem 1.

Corollary 2. If \( \Re\left(\frac{zf'(z)/f(z)}{1+zf''(z)/f'(z)}\right) > 1/2 \) for \( z \in \Delta \), then \( f \in S^*(1/2) \).

Proof. This follows from Corollary 1 upon noting that for any complex value \( w \), \( |w - 1| < 1 \iff \Re(1/w) > 1/2 \).

We next give a partial converse to Corollary 1.

Theorem 2. If \( f \in S^*(1/2) \), then \( |\left(\frac{1+zf''(z)/f'(z)}{zf''(z)/f'(z)}\right) - 1| < 1 \) for \( |z| < (2\sqrt{3} - 3)^{1/2} = 0.68 \ldots \). The result is sharp.

Proof. Set \( p(z) = zf'(z)/f(z) = 1/(1 - \omega(z)) \), where \( \omega(z) \) is a Schwarz function. We need to find the largest disk \( |z| < R \) for which \( |zp'(z)/p(z)^2| = |z\omega'(z)| < 1 \). Dieudonné [2] found the region of values for the derivative of Schwarz functions. This led to the sharp bound [3],

\[
|\omega'(z)| \leq \begin{cases} 1, & r = |z| \leq \sqrt{2} - 1 \\ \frac{(1 + r^2)^2}{4r(1 - r^2)}, & r \geq \sqrt{2} - 1. \end{cases}
\]

Since \( |z\omega'(z)| \leq (1 + r^2)/4(1 - r^2) = 1 \) for \( r = (2\sqrt{3} - 3)^{1/2} \), the proof is complete.

3. A counterexample. The extreme points of the closed convex hull of convex functions and functions starlike of order 1/2 are identical. See [1]. Since \( G_1 \subset S^*(1/2) \), one might also expect to have \( G_1 \subset K \). Surprisingly, this is not the case. We now construct a function \( f \in G_1 - K \).
THEOREM 3. $G_1 \not\subset K$.

**Proof.** $G_1 \subset S^+(1/2)$. Any of $f \in G_1$ satisfies $zf'(z)/f(z) = 1/(1 - \omega(z))$ for some Schwarz function $\omega(z)$. Setting $\alpha = 1/2$ in (2.3), we see that $f \in G_1 \iff |z\omega'(z)| < 1$ for $z \in \Delta$, which means that $z\omega'(z)$ must, also, be a Schwarz function. Since $1 + zf''(z)/f(z) = (1 + z\omega'(z))/(1 - \omega(z))$, it suffices to construct a Schwarz function $\Omega(z) = z\omega'(z)$ for which

$$\Re\left\{\frac{1 + \Omega(z)}{1 - \omega(z)}\right\} < 0$$

(3.1)

at some point $z \in \Delta$. Let

$$A = \{z \in \Delta : |z - z_0| < 10^{-5}, z_0 = e^{\pi i/4} = e^{i\theta_0}\},$$

(3.2)

and set

$$\phi(z) = (z_0 + \bar{z}_0)\left[(1 - \bar{z}_0z)^{1/N} - 1\right],$$

(3.3)

where $N$ is large enough so that $|\phi(z)/z| < 10^{-4}$ for $z \in \Delta - A$ and $|\Im \phi(z)| < 10^{-8}$ for $z \in A$. Define $\Omega$ by $\Omega(z) = 0.9999(z + \phi(z))$.

We first show that $\Omega(z)$ (and, consequently, $\omega(z)$) is a Schwarz function and then show that inequality (3.1) holds when $z = z_0$.

If

$$z \in \Delta - A,$$

(3.4)

then

$$|\Omega(z)| \leq 0.9999(|z| + |\phi(z)|) \leq 0.9999(1.0001) < 1.$$  

(3.5)

If $z \in A$, set $z = z_0 - e^\theta e^{i\phi}$, $0 < \phi < 10^{-5}$, and note that $-2\cos \theta_0 \leq \Re \phi(z) \leq 0$. If $\Re (z + \phi(z)) \geq 0$, then $|z + \Re \phi(z)| \leq |z| < 1$. If $\Re (z + \phi(z)) < 0$, then

$$|z + \Re \phi(z)| \leq \sqrt{(\cos \theta_0 + \epsilon)^2 + (\sin \theta_0 + \epsilon)^2} < \sqrt{1 + 4 \epsilon} < 1 + 2 \epsilon < 1.0001.$$  

(3.6)

Thus, if $z \in A$,

$$|\Omega(z)| \leq 0.9999|z + \Re \phi(z)| + |\Im \phi(z)| < 0.9999(1.0001) + 10^{-8} = 1.$$  

(3.7)

Therefore, $\Omega(z)$ is a Schwarz function.

We now show that (3.1) holds at $z = z_0$ for this choice of $\Omega(z)$. Since

$$\left|\frac{\Omega(z)}{z} - 1\right| = |\omega'(z) - 1| < 0.0002 \quad \text{for} \quad z \in \Delta - A,$$

(3.8)

we may write $\omega(z) = z + \eta(z)$, where $|\eta(z)| < 0.0003$ for $z \in A$. Note that

$$\left(1 - \omega(z_0)\right)^2 \Re\left\{\frac{1 + \Omega(z_0)}{1 - \omega(z_0)}\right\} = \Re\left\{(1 - \Omega(z_0))(1 + \omega(z_0))\right\}$$

$$= \Re\left\{(1 - 0.9999\bar{z}_0)(1 - \bar{z}_0 - \eta(z_0))\right\}$$

$$\leq 1 - 1.9999\cos \theta_0 + 0.9999\cos 2\theta_0 + 2|\eta(z_0)|$$

$$< 1 - 1.9999\cos (\pi/4) + 0.0006 < 0.$$  

(3.9)

Hence, the function $f$ for which $1 + zf''(z)/f'(z) = (1 + \Omega(z))/(1 - \omega(z))$ must be in $G_1 - K$. \qed
4. Convexity. Since $G_1 \notin K$, we can ask if $G_b \subset K$ for some $b < 1$. In general, $S^*(\alpha) \notin K$ even for $\alpha$ arbitrary close to 1 ($b$ close to 0). To see this, we note that $f_n(z) = z + a_n z^n$ is in $S^*(\alpha)$ if and only if $|a_n| \leq (1-\alpha)/(n-\alpha)$ and $f_n(z) \in K$ if and only if $|a_n| \leq 1/n^2$. Thus, $f(z) = z + (1-\alpha)/(n-\alpha) z^n \in S^*(\alpha) - K$ for $n > 2/(1-\alpha)$.

We next show that there are values of $b$ for which the functions in $G_b$ must be convex.

**Theorem 4.** $G_b \subset K$ for $b \leq \sqrt{2}/2$.

**Proof.** Since $f \in G_b \subset G_1 \subset S^*(1/2)$, we may write $z f'(z)/f(z) = 1/(1-\omega(z))$, where $\omega$ is a Schwarz function. For $f \in G_b$, we take $\alpha = 1/2$ in (2.3) to obtain $|z \omega'(z)| < \sqrt{2}/2$ and, consequently, $|\omega(z)| < \sqrt{2}/2$, $z \in \Delta$. We need to show that

$$\text{Re} \left\{ 1 + z f''(z)/f'(z) \right\} = \text{Re} \left\{ \frac{(1 + z \omega'(z))}{(1 - \omega(z))} \right\} > 0.$$ (4.1)

Since

$$\left| \arg \left( \frac{1 + z \omega'(z)}{1 - \omega(z)} \right) \right| \leq \left| \arg (1 + z \omega'(z)) \right| + \left| \arg (1 - \omega(z)) \right| \leq \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2},$$ (4.2)

the result follows.

In [6], MacGregor found the radius of convexity for $S^*(1/2)$ to be $(2\sqrt{3} - 3)^{1/2} = 0.68\ldots$. Since $G_1 \subset S^*(1/2)$, we know that the radius of convexity is at least this large. The following consequence of Theorem 4 is that functions in $G_1$ are convex in the disk $|z| < \sqrt{2}/2$.

**Corollary.** If $f \in G_b$, $\sqrt{2}/2 \leq b \leq 1$, then $f$ is convex in the disk $|z| < \sqrt{2}/2b$.

**Proof.** If $|z \omega'(z)| < 1$ for $z \in \Delta$, then $|z \omega'(z)| < t$ for $|z| < t < 1$. If $f \in G_b$, then $|z \omega'(z)| < b$ for $z \in \Delta$. Hence, $|z \omega'(z)| < \sqrt{2}/2$ when $|z| < \sqrt{2}/2b$.

5. Examples. Theorem 1 gives a sharp order of starlikeness for $G_b$ when $0 < b \leq 1$, with $G_1 \subset S^*(1/2)$. Our methods do not extend to $b > 1$, but we expect the order of starlikeness to decrease from $1/2$ to 0 as $b$ increases from 1 to some value $b_0$ after which functions in $G_b$ need not be starlike. We do not have a sharp result for $b > 1$, but our next example shows that the univalent functions in $G_b$ are not necessarily starlike for $b \geq 11.66$.

The function $h(z) = z(1-iz)^{i-1}$ is spiral-like [11] and, hence, in $S$ because

$$\text{Re} \left\{ e^{\pi i/4} \frac{zh'(z)}{h(z)} \right\} = \frac{1}{\sqrt{2}} \left( 1 - |z|^2 \right) > 0, \quad z \in \Delta.$$ (5.1)

Since $zh'(z)/h(z) = (1+z)/(1-iz)$, we see that $h$ is not starlike for $|z| < a$, $\sqrt{2}/2 < a < 1$. Thus, $f(z) = f_a(z) = h(az)/a$ is not starlike for $z \in \Delta$. Setting $p(z) = zf'(z)/f(z) = (1+az)/(1-az)$, we have

$$\left| \frac{zp'(z)}{(p(z))^2} \right| = \left| \frac{(1+iz)(1+az)^2}{(1-az)^2} \right| \leq \frac{\sqrt{2}a}{(1-a)^2} < 11.66$$ (5.2)
for a sufficiently close to $\sqrt{2}/2$. Hence, $f \in G_b - S^*(0)$ for $b = 11.66$.

Finally, we show that the functions in $G_b$ need not be univalent. In [5], it is shown for $h(z) = z(1 - iz)^{i-1}$ that $g(z) = \int_0^z \frac{h(t)}{t} \, dt = (1 - iz)^i - 1$ is not in $S$ because $g(z_0) = g(-z_0)$ for $z_0 = i(e^{2\pi} - 1)/(e^{2\pi} + 1), |z_0| = 0.996 ...$. We, thus, conclude that for $f(z) = g(cz)/c, c = 0.997, f \in G_b - S$ for $b$ sufficiently large.

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**References**


Silverman: Department of Mathematics, University of Charleston, Charleston, SC 29424, USA