TOTALLY REAL SUBMANIFOLDS IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. In this paper, we establish the following result: Let M be an n-dimensional complete totally real minimal submanifold immersed in CP^n with Ricci curvature bounded from below. Then either M is totally geodesic or \( \inf r \leq \frac{(3n + 1)(n - 2)}{3} \), where r is the scalar curvature of M.

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1. Introduction. Let \( CP^n \) be the n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature \( c = 4 \) and let M be an n-dimensional totally real submanifold of \( CP^n \). Let r be the scalar curvature of M. If M is compact, then many authors studied them and obtained many beautiful results (for example [2, 4, 5]).

In this paper, we make use of Yau’s maximum principle to study the complete totally real minimal submanifold with Ricci curvature bounded from below and obtain the following result.

THEOREM 1. Let M be an n-dimensional complete totally real minimal manifold immersed in \( CP^n \) with Ricci curvature bounded from below. Then either M is totally geodesic or \( \inf r \leq \frac{(3n + 1)(n - 2)}{3} \).

2. Preliminaries. Let M be an n-dimensional totally real minimal submanifold of \( CP^n \). We choose a local field of orthonormal frames \( e_1, \ldots, e_n, e_1^* = Je_1, \ldots, e_n^* = Je_n \) (J is the complex structure of \( CP^n \)), such that, restricted to M, the vectors \( e_1, \ldots, e_n \) are tangent to M. We make use of the following convention on the range of indices

\[
A, B, C, \ldots = 1, \ldots, n, 1^*, \ldots, n^*; \quad i, j, k, \ldots = 1, \ldots, n.
\] (2.1)

With respect to the frame field of \( CP^n \), let \( w^A \) be the field of dual frames. Then the structure equations of \( CP^n \) are given by

\[
dw^A = -\sum w_B^A \wedge w^B, \quad w_A^B + w_B^A = 0,
\] (2.2)

\[
dw_B^A = -\sum w_C^B \wedge w_C^A + \frac{1}{2} \sum R_{BCD}^A w_C^D \wedge w^D,
\] (2.3)

\[
R_{BCD}^A = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} + J_{AC} J_{BD} - J_{AD} J_{BC} + 2 J_{AB} J_{CD},
\] (2.4)
where $J = J_{AB}e_A \otimes e_B$, so that

$$(J_{AB}) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

(2.5)

where $I_n$ is the identity matrix of order $n$. We restrict these forms to $M$. Then from [2], we have

$$w^{i*} = 0, \quad w^I_j = w^i_j^*, \quad w^i_j = w^I_j,$$

(2.6)

$$w^k_i = \sum h^k_{ij} w^j, \quad h^k_{ij} = h^k_{ji} = h^k_{ik},$$

(2.7)

$$dw^I_j = -\sum w^i_j \wedge w^j_i, \quad w^i_j = 0,$$

(2.8)

$$R^I_{jkl} = \tilde{R}^I_{jkl} k^k + \sum (h^m_{ik} h^m_{ji} - h^m_{il} h^m_{jk}),$$

(2.10)

$$Rw^i_j = -\sum (h^s_{ik} \wedge w^s_i) + \frac{1}{2} \sum R^s_{ijk} k^i \wedge w^j,$$

(2.11)

$$R^s_{jkl} = \tilde{R}^s_{jkl} + \sum (h^t_{km} h^t_{ml} - h^t_{ml} h^t_{km}).$$

(2.12)

The second fundamental form $h$ of $M$ in $\mathbb{C}P^n$ is defined as $h = \sum h^{i*}_{ij} w^i \otimes e^*_j$, whose squared length is $\|h\|^2 = \sum (h^{i*}_{ij})^2$.

If $M$ is minimal in $\mathbb{C}P^n$, i.e., trace $h = 0$, then from (2.4) and (2.10), we have

$$r = n(n - 1) - \|h\|^2,$$

(2.13)

where $r$ is the scalar curvature of $M$.

Define $h^{m*}_{ij}$ and $h^{m*}_{ijkl}$ by

$$\sum h^{m*}_{ij} w^k = dh^{m*}_{ij} - \sum h^{m*}_{kj} w^k - \sum h^{m*}_{ik} w^k + \sum h^{m*}_{ij} w^m_{ik},$$

(2.14)

$$\sum h^{m*}_{ijkl} w^i = dh^{m*}_{ijkl} - \sum h^{m*}_{ijlk} w^i - \sum h^{m*}_{iklj} w^i - \sum h^{m*}_{ijkl} w^i + \sum h^{m*}_{ijkl} w^m_{ij},$$

(2.15)

respectively.

Let $H^{i*}$ and $\Delta$ denote the $(n \times n)$-matrix $(h^{i*}_{ij})$ and the Laplacian on $M$, respectively. By a simple calculation, we have (cf. [2])

$$\frac{1}{2} \Delta \|h\|^2 = \sum (h^{i*}_{ik})^2 + (n + 1) \|h\|^2 + \sum \text{tr} \left( H^{i*} H^r (\text{tr} H^{i*} H^r) \right)^2$$

$$- \sum \left( \text{tr} H^{i*} \text{tr} H^r \right)^2.$$

(2.16)

The following lemma is important in this paper.

**Lemma 1** [6]. Let $M^n$ be a complete Riemannian manifold with Ricci curvature bounded from below and let $f$ be a $C^2$-function bounded from above on $M^n$, then for all $\epsilon > 0$, there exists a point $x \in M^n$ at which

(i) $\sup f - \epsilon < f(x)$;

(ii) $\|\nabla f(x)\| < \epsilon$;

(iii) $\Delta f(x) < \epsilon$. 

PROOF OF THE MAIN THEOREM. By [3], we have \( \sum (\text{tr} H_i^* H_j^*)^2 = \sum (\text{tr} H_i^2)^2 \). From [1], we know that \( \sum \text{tr} (H_i^* H_j^* - H_j^* H_i^*)^2 - \sum (\text{tr} H_i^2)^2 \geq -3/2 \| h \|^4 \). So, from (2.16), we obtain \( \frac{1}{2} \Delta \| h \|^2 \geq \| h \|^2 (n + 1) - 3/2 \| h \|^2 \). (2.17)

We know that \( \| h \|^2 = n(n - 1) - r \). By the condition of the theorem, we conclude that \( \| h \|^2 \) is bounded. We define \( f = \| h \|^2 \) and \( F = (f + a)^{1/2} \) (where \( a > 0 \) is any positive constant number). \( F \) is bounded. We have

\[
dF = \frac{1}{2} (f + a)^{-1/2} df, \\
\Delta F = \frac{1}{2} (f + a)^{-3/2} \| df \|^2 + (f + a)^{-1/2} \Delta f \\
= \frac{1}{2} (\| df \|^2 + \Delta f) (f + a)^{-1/2},
\]

i.e.,

\[
\Delta F = \frac{1}{2F} (\| df \|^2 + \Delta f).
\]

Hence, \( F \Delta F = -\| df \|^2 + 1/2 \Delta f \) or \( 1/2 \Delta f = F \Delta F + \| df \|^2 \).

Applying Lemma 1 to \( F \), we have for all \( \epsilon > 0 \), there exists a point \( x \in M \) such that at \( x \)

\[
\| dF(x) < \epsilon \|; \\
\Delta F(x) < \epsilon; \\
F(x) > \sup F - \epsilon.
\]

From (2.21), (2.22), and (2.23), we have

\[
\frac{1}{2} \Delta f < \epsilon^2 + F \epsilon = \epsilon (\epsilon + F). 
\]

We take a sequence \( \{ \epsilon_m \} \) such that \( \epsilon_m \to 0 (m \to \infty) \) and for all \( m \), there exists a point \( x_m \in M \) such that (2.21), (2.22), and (2.23) hold. Therefore, \( \epsilon_m (\epsilon_m + F(x_m)) \to 0 (m \to \infty) \) (because \( F \) is bounded).

From (2.23), we have \( F(x_m) > \sup F - \epsilon_m \). Because \( \{ F(x_m) \} \) is a bounded sequence. So we get \( F(x_m) \to F_0 \) (if necessary, we can choose a subsequence). Hence, \( F_0 \geq \sup F \). So we have

\[
F_0 = \sup F.
\]

From the definition of \( F \), we get

\[
f(x_m) \to f = \sup f.
\]

(2.17) and (2.24) imply that

\[
f \left( (n + 1) - \frac{3}{2} f \right) \leq \frac{1}{2} \Delta f \leq \epsilon (\epsilon + F), 
\]

and

\[
f(x_m) \left( (n + 1) - \frac{3}{2} f(x_m) \right) < \epsilon_m^2 + \epsilon_m F(x_m) \leq \epsilon_m^2 + \epsilon_m F_0
\]

(2.28)
let $m \to \infty$, then $\epsilon_m \to 0$ and $f(x_m) \to f_0$. Hence,

$$f_0((n+1)-\frac{3}{2}f_0) \leq 0. \quad (2.29)$$

(i) if $f_0 = 0$, we have $f = \|h\|^2 = 0$. Hence, $M$ is totally geodesic.

(ii) if $f_0 > 0$, we have $(n+1)-3/2f_0 \leq 0$ and $f_0 \geq 2/3(n+1)$, that is, $\sup \|h\|^2 \geq 2/3(n+1)$. Therefore, $\inf r \leq (3n+1)(n-2)/3$. This completes the proof. \qed

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References


