THE NEUTRIX CONVOLUTION PRODUCT IN $\mathcal{D}'(\mathbb{R}^m)$ AND THE EXCHANGE FORMULA

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ABSTRACT. One of the problems in distribution theory is the lack of definition for convolutions and products of distributions in general. In quantum theory and physics (see e.g. [1] and [2]), one finds that some convolutions and products such as $\frac{1}{2} \cdot \delta$ are in use. In [3], a definition for product of distributions and some results of products are given using a specific delta sequence $\delta_n(z) = n^2 \rho(n^2 z^2)$ in an $m$-dimensional space. In this paper, we use the Fourier transform on $D'(\mathbb{R}^m)$ and the exchange formula to define convolutions of ultradistributions in $Z'(\mathbb{R}^m)$ in terms of products of distributions in $D'(\mathbb{R}^m)$. We prove a theorem which states that for arbitrary elements $f$ and $g$ in $Z'(\mathbb{R}^m)$, the neutrix convolution $f \otimes g$ exists in $Z'(\mathbb{R}^m)$ if and only if the product $f \circ g$ exists in $D'(\mathbb{R}^m)$. Some results of convolutions are obtained by employing the neutrix calculus given by van der Corput [4].

KEY WORDS AND PHRASES: Distributions, delta sequence, neutrix limit, convolution.

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1. INTRODUCTION

In the following, let $\rho(x)$ be a fixed infinitely differentiable function with the properties

(i) $\rho(x) = 0$, $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)dx = 1$.

We define the function $\delta_n(x)$ by $\delta_n(x) = n \rho(nx)$ for $n = 1, 2, \cdots$. It is clear that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta$.

Now let $D$ be the space of infinitely differentiable functions with compact support. If $f$ is an arbitrary distribution in $D'$, we define the function $f_n$ by $f_n = f \ast \delta_n$. It follows that $\{f_n\}$ is a sequence of infinitely differentiable functions converging to $f$.

The following definition was given by B. Fisher [5].

DEFINITION 1. Let $f$ and $g$ be distributions in $D'$ and let $g_n = g \ast \delta_n$. We say that the neutrix product $f \circ g$ of $f$ and $g$ exists and equals $h$ if

$$N - \lim_{n \to \infty} (fg_n, \phi) = (h, \phi)$$

for all $\phi$ in $D$, where $N$ is the neutrix (see van der Corput [4]) having domain $N' = \{1, 2, \cdots, n, \cdots\}$ and range $N''$ the real numbers with negligible functions finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n, \quad \ln^{k} n (\lambda > 0, r = 1, 2, \cdots)$$

and all functions of $n$ which converge to zero as $n$ tends to infinity.
Let $D'(m)$ be the space of distributions defined on the space $D(m)$ of infinitely differentiable functions of the variable $x = (x_1, x_2, \ldots, x_m)$ with compact support.

In order to give a definition for the neutrix product $f \circ g$ of two distributions $f$ and $g$ in $D'(m)$, we attempt to define a $\delta$-sequence in $D(m)$ by putting

$$
\delta_n(x_1, x_2, \ldots, x_m) = \delta_n(x_1) \cdots \delta_n(x_m),
$$

where $\delta_n$ is defined as above. However, this definition is very difficult to use for distributions in $D'(m)$ which are functions of $r$, where $r = (x_1^2 + \cdots + x_m^2)^{1/2}$. Therefore an alternative definition was introduced in [3].

From now on we let $\rho(s)$ be a fixed infinitely differentiable function defined on $R^+ = [0, \infty)$ having the properties

(i) $\rho(s) = 0$, $s \geq 1$, 
(ii) $\rho(s) \geq 0$.

Define the function $\delta_n(x)$, with $x \in R^m$, by

$$
\delta_n(x) = C_m n^m \rho(n^2 r^2)
$$

for $n = 1, 2, \ldots$, where $C_m$ is a constant such that

$$
\int_{R^m} \delta_n(x) dx = 1.
$$

**DEFINITION 2.** Let $f$ and $g$ be distributions in $D'(m)$ and let

$$
g_n(x) = (g \ast \delta_n)(x) = (g(x - t), \delta_n(t))
$$

where $t = (t_1, t_2, \ldots, t_m)$. We say that the neutrix product $f \circ g$ of $f$ and $g$ exists and is equal to $h$ on the open interval $(a, b)$, where $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$, if

$$
N - \lim_{n \to \infty} (fg_n, \phi) = (h, \phi)
$$

for all test functions $\phi$ is $D(m)$ with support contained in the interval $(a, b)$.

2. **FOURIER TRANSFORM ON $D'(m)$**

As in Gel'fand and Shilov [6], we define the Fourier transform of a function $\phi$ in $D(m)$ by

$$
F(\phi)(s) = \psi(s) = \int_{R^m} \phi(x)e^{ix \cdot s} dx,
$$

where $(x, s)$ denotes $x_1 s_1 + \cdots + x_m s_m$.

The bounded support of $\phi(x)$ makes it possible for $\psi$ to be continued to complex values of its argument $s = (s_1, \ldots, s_m) = (s_1 + i\tau_1, \ldots, s_n + i\tau_m)$:

$$
\psi(s) = \int_{R^m} \phi(x)e^{ix \cdot s} dx.
$$

Our new function $\psi(s)$, defined on $C^m$, in the space of functions of $m$ complex variables, is continuous and analytic in each of its variable $s_k$. If $\phi(x)$ vanishes for $|x_k| > a_k$, $k = 1, \ldots, m$, then $\psi(s)$ satisfies the inequality

$$
|\phi^{(s_1, \ldots, s_m)}|(s_1 + i\tau_1, \ldots, s_m + i\tau_m)| \leq C_q \exp(a_1|\tau_1| + \cdots + a_m|\tau_m|).
$$

Conversely, every entire function $\psi(s_1, \ldots, s_m)$ satisfying the above inequality is the Fourier transform of some $\phi(x_1, \ldots, x_m)$ in $D(m)$ which vanishes for $|x_k| > a_k$, $k = 1, 2, \ldots, m$.

The set of all entire analytic functions $Z(m)$ with the property (1) is in fact the space

$$
F(D(m)) = \{ \psi : \exists \phi \in D(m) \text{ such that } F(\phi) = \psi \}.
Convergence in $Z(m)$ is defined via convergence in $D(m)$: a sequence $\{\psi_n\}$ tends to zero in $Z(m)$ if the sequence $\{f_n\}$ tends to zero in $D(m)$, where $F(f_n) = \psi_n$. The Fourier transform $\tilde{f}$ of a distribution in $D'(m)$ is an ultradistribution in $Z'(m)$, i.e., a continuous linear functional on $Z(m)$. It is defined by Parseval's equation

$$\langle \tilde{f}, \phi \rangle = 2\pi \int_{-\infty}^{\infty} \psi(\sigma) d\sigma,$$

where $\psi = \tilde{\phi}$.

Therefore $\{\tau_n\}$ is a sequence in $Z(m) \subset Z'(m)$ converging to $1$ in $Z'(m)$.

Now let $\tilde{f}$ be an arbitrary ultradistribution in $Z'(m)$. Then there exists a distribution $f$ in $D'(m)$ such that $\tilde{f} = F(f)$. Setting $\tilde{f}_n = F(f_n \delta_n) = F(f_n)$, we have

$$\langle \tilde{f}_n, \psi \rangle = 2\pi \int_{-\infty}^{\infty} \psi(\sigma) d\sigma = (1, \psi)$$

where $\psi = \tilde{\phi}$ in $Z(m)$.

**Lemma 1.** Let $\tilde{g}$ be an arbitrary ultradistribution in $Z'(m)$ and let $g_n = F(g \delta_n)$. Then the function

$$\Theta_n(\nu) = (g_n(\sigma), \psi(\sigma + \nu))$$

is in $Z(m)$ for all $\psi$ in $Z(m)$.

Indeed,

$$\Theta_n(\nu) = \langle F(g_n), F(e^{i\pi \nu \phi(x)})(\sigma) \rangle = 2\pi \int_{-\infty}^{\infty} \psi(\sigma + \nu) d\sigma$$

Now the result of the lemma follows on noting that $g_n \phi$ is in $D(m)$.

We now modify the definition for the convolution product of two distributions in $D'(m)$ given by Gel'fand and Shilov with

**Definition 3.** Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$ such that the function $(\tilde{g}(\sigma), \psi(\sigma + \nu))$ is in $Z(m)$ for all $\psi$ in $Z(m)$. Then the convolution product $\tilde{f} * \tilde{g}$ is defined by

$$\langle ((\tilde{f} * \tilde{g})(\sigma), \psi(\sigma)) \rangle = \langle \tilde{f}(\nu), (\tilde{g}(\sigma), \psi(\sigma + \nu)) \rangle$$

for all $\psi$ in $Z(m)$.

It follows that $\tilde{f} * \tilde{g}$ exists if $g \phi$ is in $D(m)$. (This condition is not always true for all $g$ in $D'(m)$. If $\tilde{g} \in Z(m)$, then $g \phi \in D(m)$.) Indeed

$$\langle \tilde{g}(\sigma), \psi(\sigma + \nu) \rangle = 2\pi \int_{-\infty}^{\infty} \psi(\sigma + \nu) d\sigma = 2\pi F(g \phi)(\nu),$$

where $\tilde{g} = F(g)$ and $\psi = F(\phi)$.

The following theorem then holds:

**Theorem 1.** Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$ and suppose that the convolution product $\tilde{f} * \tilde{g}$ exists. Then
\[(\tilde{f} \ast \tilde{g})' = \tilde{f} \ast \tilde{g}', \quad (2)\]
\[(\tilde{f} \ast \tilde{g})' = \tilde{f}' \ast \tilde{g}. \quad (3)\]

**PROOF.** If \(F(\phi) = \psi\), we have
\[\psi'(\sigma) = F(\text{ix}\phi(x))(\sigma).\]

Hence \(Z'(m)\) is closed under differentiation.

Certainly
\[
\left( (\tilde{f} \ast \tilde{g})', \psi \right) = - (\tilde{f} \ast \tilde{g}, \psi') = - (\tilde{f}(\nu), (\tilde{g}(\sigma), \psi'(\sigma + \nu)))
\]
\[
= (\tilde{f}(\nu), (\tilde{g}'(\sigma), \psi(\sigma + \nu))) = (\tilde{f} \ast \tilde{g}', \psi)
\]
for all \(\psi\) in \(Z(m)\). Equation (2) follows.

From the fact that if \(F(\phi)\), we get
\[\psi'(\sigma + \nu) = F(\text{ix}\phi(x)e^{\text{iz}\nu})(\sigma).
\]

It follows that
\[
(\tilde{g}(\sigma), \psi'(\sigma + \nu)) = 2\pi (g(x), \text{ix}\phi(x)e^{\text{iz}\nu})
\]
\[
= 2\pi \frac{d}{d\nu} (g(x), \phi(x)e^{\text{iz}\nu})
\]
\[
= \frac{d}{d\nu} (\tilde{g}(\sigma), \psi(\sigma + \nu)).
\]

Hence
\[
\left( (\tilde{f} \ast \tilde{g})', \psi \right) = \left( \tilde{f}'(\nu), (\tilde{g}(\sigma), \psi(\sigma + \nu)) \right) = \left( \tilde{f} \ast \tilde{g}', \psi \right)
\]
for all \(\psi\) in \(Z(m)\) and Equation (3) follows.

Note that \(\tilde{f}' \neq F(f')\) is general.

We now note that if \(\tilde{f}\) and \(\tilde{g}\) are arbitrary ultradistributions in \(Z'(m)\), then the convolution product \(\tilde{f} \ast \tilde{g}_n\) always exists by the above definition (3) since by Lemma 1, \((\tilde{g}_n(\sigma), \psi(\sigma + \nu))\) in \(Z(m)\) for all \(\psi\) in \(Z(m)\). This leads us to the following definition.

**DEFINITION 4.** Let \(\tilde{f}\) and \(\tilde{g}\) be ultradistributions in \(Z'(m)\) and let \(\tilde{g}_n = \tilde{g}_0 \ast \tilde{g}_n\). Then the neutrix convolution product \(\tilde{f} \otimes \tilde{g}\) is defined to be the neutrix limit of the sequence \(\{\tilde{f} \ast \tilde{g}_n\}\), provided the neutrix limit \(\tilde{h}\) exists in the sense that
\[N - \lim_{n \to \infty} (\tilde{f} \ast \tilde{g}_n, \psi) = (\tilde{h}, \psi) \quad \text{for all } \psi \text{ in } Z(m),\]

Definition 4 is indeed a generalization of Definition 3, since if the convolution product \(\tilde{f} \ast \tilde{g}\) exists by Definition 3, then \((\tilde{g}(\sigma), \psi(\sigma + \nu)) \in Z(m)\), i.e., \(g \phi \in D(m)\) for all \(\phi \in D(m)\). This implies \(g \in C^\infty(m)\).

Therefore \((\tilde{g}_n(\sigma), \psi(\sigma + \nu)) = 2\pi F(\phi(\nu))\) converges to \((\tilde{g}(\sigma), \psi(\sigma + \nu))\) in \(Z(m)\). This is because \(g_n \phi \to \phi\) (if \(f \in C^\infty\), then \(f_n \phi\) (where \(f_n = f \ast \delta_n\) converges to \(f\phi\) uniformly on the support of \(\phi\) in \(D(m)\), and \(N - \lim_{n \to \infty} (\tilde{f} \ast \tilde{g}_n, \psi) = (\tilde{f} \ast \tilde{g}, \psi)\) for all \(\psi \) in \(Z(m)\).

The following theorem holds for the neutrix convolution product.

**THEOREM 2.** Let \(\tilde{f}\) and \(\tilde{g}\) be ultradistributions in \(Z'(m)\) and suppose that their neutrix convolution product exists. Then the neutrix convolution product \(\tilde{f} \otimes \tilde{g}\) exists and
\[(\tilde{f} \otimes \tilde{g})' = \tilde{f}' \otimes \tilde{g}.\]

**PROOF.** We have
\[
\left( (\tilde{f} \ast \tilde{g}_n)', \psi \right) = \left( \tilde{f}' \ast \tilde{g}_n, \psi \right) = - (\tilde{f} \ast \tilde{g}_n, \psi')
\]
and it follows that
for arbitrary $\psi$ in $Z(m)$. The result of the theorem follows.

Note that $(\tilde{f} \otimes \tilde{g})' = \tilde{f} \otimes \tilde{g}'$ iff $N - \lim_{n \to \infty} (\tilde{f} * (\tilde{g} r_n), \psi) = 0$ for all $\psi$ in $Z(m)$.

We now prove our main result, the exchange formula.

**THEOREM 3.** Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$. Then the neutrix convolution product $\tilde{f} \otimes \tilde{g}$ exists in $Z'(m)$ iff the neutrix product $f \circ g$ exists in $D'(m)$ and the exchange formula

\[ \tilde{f} \otimes \tilde{g} = 2\pi F(f \circ g) \]

is then satisfied.

**PROOF.** Let $\psi = F(\phi)$ be an arbitrary function in $Z(m)$ and let

\[ \Theta_n(\nu) = (\tilde{g}_n(\sigma), (\psi(\sigma + \nu)) = 2\pi F(g_n(\phi))(\nu). \]

Then on using Parseval's equation we have

\[ (\tilde{f}(\nu), \Theta_n(\nu)) = 2\pi (\tilde{f}(\nu), F(g_n(\phi))(\nu)) = (2\pi)^2 (f g_n, \phi). \]

If the neutrix convolution product $\tilde{f} \otimes \tilde{g}$ exists then

\[ (\tilde{f} \otimes \tilde{g}, \phi) = N - \lim_{n \to \infty} (\tilde{f}(\nu), \Theta_n(\nu)) = (2\pi)^2 N - \lim_{n \to \infty} (f g_n, \phi) \]

\[ = (2\pi)^2 (f \circ g, \phi) = 2\pi F(f \circ g, F(\phi)). \]

The neutrix product $f \circ g$ therefore exists and the exchange formula is satisfied.

Conversely, the existence of the neutrix product $f \circ g$ implies the existence of the neutrix convolution product and the exchange formula.

4. **SOME RESULTS**

The following Fourier transforms of the functions $r^\lambda$ and $\Delta^k \delta(x)$ were given in [6]

\[ F(r^\lambda) = 2^{\lambda+m} \pi^{m/2} \Gamma\left(\frac{\lambda+m}{2}\right) \rho^{-\lambda-m} \]

where $\lambda \neq -m, -m - 2, \ldots$ and $\rho = \sqrt{\sum_{i=1}^{m} \sigma_i^2}$, and

\[ F[P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right) f(x)] = P(-is_1, \ldots, -is_m) F(f). \]

Hence it follows that

\[ F(\Delta^k \delta(x)) = \rho^{2k} F(\delta) = \rho^{2k}, \]

where $\Delta$ denotes the Laplace operator.

**THEOREM 4.** The neutrix convolution products $\rho^{2k-m} \otimes 1$ and $\rho^{2k-1-m} \otimes 1$ exist and

\[ \rho^{2k-m} \otimes 1 = \frac{\Gamma(k)2^{k-m+1}\rho^{2k}}{\Gamma\left(\frac{m-2k}{2}\right) \pi^{m/2-1}k!m(m+2)\cdots(m+2k-2)} \]

for $k = 1, 2, \ldots, \left[\frac{m-1}{2}\right]$ and

\[ \rho^{2k-1-m} \otimes 1 = 0 \]

for $k = 1, 2, \ldots, \left[\frac{m}{2}\right]$.

**PROOF.** We have the following neutrix product (see [3]),
By the exchange formula
\[ r^{-2k} \cdot \delta(x) = \frac{\Delta^k \delta(x)}{2^k k!m(m+2)\cdots(m+2k-2)} \]
for \( k = 1, 2, \ldots, \left[ \frac{m-1}{2} \right] \) and
\[ r^{1-2k} \cdot \delta(x) = 0 \]
for \( k = 1, 2, \ldots, \left[ \frac{m}{2} \right] \).

By the exchange formula
\[
F(r^{-2k}) \otimes F(\delta) = 2\pi F(r^{-2k} \cdot \delta)
= 2\pi \frac{F(\Delta^k \delta)}{2^k k!m(m+2)\cdots(m+2k-2)}
= 2\pi \frac{\rho^{2k}}{2^k k!m(m+2)\cdots(m+2k-2)}.
\]

Thus
\[ 2^{-2k+m} \pi^{m/2} \frac{\Gamma\left(\frac{m-2}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \rho^{2k-m} \otimes 1 = \frac{2\pi \rho^{2k}}{2^k k!m(m+2)\cdots(m+2k-2)}.
\]

It follows that
\[ \rho^{2k-m} \otimes 1 = \frac{\Gamma(k)2^{k-m+1} \rho^{2k}}{\Gamma\left(\frac{m-2k}{2}\right) \pi^{m/2-1} k!m(m+2)\cdots(m+2k-2)}.
\]

The second equation follows easily.

The following neutrix product is also given in [3]
\[ r^{-2k} \cdot \Delta \delta(x) = \frac{\Delta^{k+1} \delta(x)}{2^k (k+1)!(m+2)\cdots(m+2k)} \]
for \( k = 1, 2, \ldots, \left[ \frac{m-1}{2} \right] \) and
\[ r^{1-2k} \cdot \Delta \delta(x) = 0 \]
for \( k = 1, 2, \ldots, \left[ \frac{m}{2} \right] \).

Hence we obtain

**THEOREM 5.** The neutrix convolution product \( \rho^{2k-m} \otimes \rho^2 \) and \( \rho^{2k-1-m} \otimes \rho^2 \) exist and
\[ \rho^{2k-m} \otimes \rho^2 = \frac{\Gamma(k)2^{k-m+1}}{\Gamma\left(\frac{m-2k}{2}\right) \pi^{m/2-1} (k+1)!(m+2)\cdots(m+2k)} \]
for \( k = 1, 2, \ldots, \left[ \frac{m-1}{2} \right] \) and
\[ \rho^{2k-1-m} \otimes \rho^2 = 0 \]
for \( k = 1, 2, \ldots, \left[ \frac{m}{2} \right] \).

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