GENERALIZED FRACTIONAL CALCULUS TO A SUBCLASS OF ANALYTIC FUNCTIONS FOR OPERATORS ON HILBERT SPACE

YONG CHAN KIM, JAE HO CHOI, AND JIN SEOP LEE

Department of Mathematics
Yeungnam University
Gyongsan 712-749, KOREA

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ABSTRACT. In this paper, we investigate some generalized results of applications of fractional integral and derivative operators to a subclass of analytic functions for operators on Hilbert space.

KEY WORDS AND PHRASES: Multivalent function, Fractional calculus, Riesz-Dunford integral.

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1. INTRODUCTION AND DEFINITIONS

Let \( A \) denote the class of functions of the form:

\[
    f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \quad (a_1 := 1),
\]

which are analytic in the open unit disk

\[
    \mathcal{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.
\]

Also let \( S \) denote the class of all functions in \( A \) which are univalent in the unit disk \( \mathcal{U} \).

Let \( S_0(\alpha, \beta, \gamma, p) \) denote the class of functions

\[
    f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_{n+p} \geq 0),
\]

which are analytic and \( p \)-valent in \( \mathcal{U} \) and satisfy the condition

\[
    \left| \frac{zf'(z)}{f(z)} - p \right| < \beta \left| \frac{zf'(z)}{f(z)} + (p - \gamma) \right|
\]

for \( 0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \gamma < p, p \in \mathbb{N} \) and \( z \in \mathcal{U} \). See Lee et al [1] for further information on them. It is easily found that \( S_0(\alpha, \beta, \gamma, p) \subset A \) when \( p = 1 \).

Let \( a, b, \) and \( c \) be complex numbers with \( c \neq 0, -1, -2, \cdots \). Then the Gaussian hypergeometric function \( \mathbf{2F_1}(z) \) is defined by

\[
    \mathbf{2F_1}(z) \equiv \mathbf{2F_1}(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},
\]

where \((\lambda)_n\) is the Pochhammer symbol defined, in terms of the Gamma function, by

\[
    (\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n = 0) \cr \lambda(\lambda + 1) \cdots (\lambda + n - 1), & (n \in \mathbb{N} := \{1, 2, 3, \cdots\}) \end{cases}.
\]
Let $A$ be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. For a complex valued function $f$ analytic on a domain $E$ of the complex plane containing the spectrum $\sigma(A)$ of $A$ we denote $f(A)$ as Riesz-Dunford integral [2, p. 568], that is,

$$f(A) := \frac{1}{2\pi i} \int_C f(z)(zI - A)^{-1}dz,$$

(1.6)

where $I$ is the identity operator on $\mathcal{H}$ and $C$ is positively oriented simple closed rectifiable contour containing $\sigma(A)$.

Also $f(A)$ can be defined by the series $f(A) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} A^n$ which converges in the norm topology [3].

Xiaopei [4] defined $S_0(\alpha, \beta, \gamma, p; A)$ by the class of functions

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_{n+p} \geq 0),$$

which is analytic and $p$-valent in $\mathcal{U}$ and satisfies the condition,

$$\|A f'(A) - pf(A)\| < b \|\alpha A f'(A) + (p - \gamma) f(A)\|$$

(1.7)

for $0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \gamma < p, p \in \mathbb{N}$ and all operators $A$ with $\|A\| < 1$ and $A \neq 0$ ($0$ denotes the zero operator on $\mathcal{H}$).

Let $A^*$ denote the conjugate operator of $A$.

**DEFINITION 1** ([4]). The fractional integral for operator of order $\alpha$ is defined by

$$D^\alpha_A f(A) = \frac{1}{\Gamma(\alpha)} \int_0^1 A^\alpha f(tA) (1-t)^{\alpha-1}dt,$$

(1.8)

where $\alpha > 0$ and $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin.

**DEFINITION 2** ([4]). The fractional derivative for operator of order $\alpha$ is defined by

$$D^\alpha_A f(A) = \frac{1}{\Gamma(1 - \alpha)} g'(A),$$

(1.9)

where $g(z) = \int_0^1 z^{1-\alpha} f(tz)(1-t)^{-\alpha}dt \ (0 < \alpha < 1)$ and $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin.

Srivastava et al. [5] introduced a fractional integral operator $I_{0,x}^{\alpha,b,c}$ defined by (cf. [6])

$$I_{0,x}^{\alpha,b,c} f(z) = \frac{z^{-\beta}}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} {}_2F_1(a+b, -c; a; 1-t) f(tz) dt$$

(1.10a)

and Owa et al. [7] studied the fractional operator $J_{0,x}^{\alpha,b,c}$ defined by (see also Kim et al. [8])

$$J_{0,x}^{\alpha,b,c} f(z) = \frac{\Gamma(2-b)\Gamma(2+a+c)}{\Gamma(2-b+c)} z^b I_{0,x}^{\alpha,b,c} f(z) \quad (f \in \mathcal{A}).$$

(1.11a)

The fractional derivative operator $D_{0,x}^{\alpha,b,c}$ is defined by (cf. [9])

$$D_{0,x}^{\alpha,b,c} f(z) = \frac{d}{dz} \left( \frac{z^{-\beta}}{\Gamma(1-a)} \int_0^1 (1-t)^{-\alpha-1} {}_2F_1(b-a+1, -c; 1-a; 1-t) f(tz) dt \right)$$

(1.12a)

(0 \leq \alpha < 1; b, c \in \mathbb{R}; f(z) \in \mathcal{A}).

And we define $D_{0,x}^{\alpha-a,b,c}$ by
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\[ D_{0, \lambda}^{\alpha, \beta, \gamma} f(z) = \frac{d^n}{dz^n} D_{0, \lambda}^{\alpha, \beta, \gamma} f(z). \]  

(1.13)

For all invertible operator \( A \), we introduce the following definition:

**DEFINITION 3.** The fractional integral operator for operator \( I_{0, A}^{\alpha, \beta, \gamma} \) is defined by

\[ I_{0, A}^{\alpha, \beta, \gamma} f(A) = \frac{1}{\Gamma(\alpha)} \int_0^1 A^{-b} \frac{2 F_1(a + b - c; a; 1 - t)}{(1 - t)^{\alpha - 1}} dt, \]

(1.14)

where \( \alpha > 0 \) and \( b, c \in \mathbb{R} \).

The fractional derivative operator for operator \( D_{0, A}^{\alpha, \beta, \gamma} \) is defined by

\[ D_{0, A}^{\alpha, \beta, \gamma} f(A) = \frac{1}{\Gamma(1 - \alpha)} g'(A), \]

(1.15)

where

\[ g(z) = \int_0^1 z^{-b} \frac{2 F_1(b - a + 1, -c; 1 - a; 1 - t)}{(1 - t)^{-\alpha}} dt, \]

0 < \( \alpha \) < 1 and \( b, c \in \mathbb{R} \). In both (1.14) and (1.15) \( f(z) \) is an analytic function in a simply-connected region of the \( z \)-plane containing the origin with the order

\[ f(z) = O(\epsilon^{|z|}), \quad z \to 0, \]

where \( \epsilon > \max\{0, b - c\} - 1 \) and the multiplicity of \((1 - t)^{-\alpha - 1}\) is in (1.14) (and that of \((1 - t)^{-\alpha}\) in (1.15)) removed by requiring \( \log(1 - t) \) to be real when \( 1 - t > 0 \).

We note that

\[ a, a - 1, c \]

\[ \frac{d^a}{dz^a} f(A) = D_{0, A}^{\alpha, \beta, \gamma} f(A) \quad \text{and} \quad D_{0, A}^{\alpha, \beta, \gamma} f(A) = D_{A}^{\alpha, \beta, \gamma} f(A). \]

(1.17)

The object of this paper is to prove the distortion theorems of fractional integral and derivative operators to \( S_0(\alpha, \beta, \gamma, p; A) \).

2. RESULTS

**LEMMA 1** (Xiaopei [4, Theorem 2.1]. An analytic function \( f(z) \) is in the class \( S_0(\alpha, \beta, \gamma, p; A) \) for all proper contraction \( A \) with \( A \neq 0 \) if and only if

\[ \sum_{k=1}^{\infty} \{k + \beta[p - \gamma + \alpha(k + p)]\} a_{k+p} \leq \beta(p - \gamma + \alpha p) \]

(2.1)

for \( 0 \leq \alpha \leq 1, \ 0 < \beta \leq 1, \ 0 \leq \gamma < p, \) and \( p \in \mathbb{N} \).

The result is sharp for the function

\[ f(z) = z^p - \frac{\beta(p - \gamma + \alpha p)}{k + \beta[p - \gamma + \alpha(k + p)]} z^{k+p} \quad (k \geq 1). \]

**THEOREM 1.** Let \( p > \max\{b - c - 1, b - 1, -1 - c - a\} \) and \( a(p + 1) > b(a + c) \). If \( f(z) \in S_0(\alpha, \beta, \gamma, p; A) \), then

\[ \| I_{0, A}^{\alpha, \beta, \gamma} f(A) \| \leq \frac{\Gamma(p + 1 - b + c)\Gamma(p + 1)}{(p + 1 - b)\Gamma(a + p + 1 + c)} \| A \|^{-b} \]

\[ \frac{\beta(p - \gamma + \alpha p)\Gamma(p + 1 - b + c)\Gamma(p + 1)}{(1 + \beta[p - \gamma + \alpha(p + 1)])\Gamma(p + 1 - b)\Gamma(a + p + 1 + c)} \| A \|^{p+1-b} \]

(2.2)

and
\[ \| r_{0,A} f(A) \|_{p,q} \geq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p+1-b)\Gamma(a+p+1+c)} \| A \|^{p-b} \frac{\beta(p-\gamma+\alpha(p+1))\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}^{\gamma}(p+1-b)\Gamma(a+p+1+c)} \| A \|^{p+1-b} \]  

(2.3)

for \( a > 0, b, c \in \mathbb{R} \) and all invertible operator \( A \) with \( (A^q)^{1/q} = A^{1/q} \) \((q \in \mathbb{N})\), \( \| A \| < 1 \) and \( r_{sp}(A)r_{sp}(A^{-1}) \leq 1 \), where \( r_{sp}(A) \) is the radius of spectrum of \( A \).

**PROOF.** Consider the function

\[ F(A) = \frac{\Gamma(p+1-b)\Gamma(a+p+1+c)}{\Gamma(p+1-b+c)\Gamma(p+1)} A^b r_{0,A} f(A) \]

\[ = A^p - \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p+1-b)\Gamma(a+p+1+c)}{\Gamma(k+p+1)\Gamma(a+k+p+1+c)\Gamma(p+1)\Gamma(p+1-b+c)} a_{k+p} A^{k+p} \]

\[ = A^p - \sum_{k=1}^{\infty} B_{k+p} A^{k+p}, \quad (2.4) \]

where

\[ B_{k+p} = \frac{\Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p+1-b)\Gamma(a+p+1+c)}{\Gamma(k+p+1)\Gamma(a+k+p+1+c)\Gamma(p+1)\Gamma(p+1-b+c)} a_{k+p}. \]

Hence, for convenience, we put

\[ \Phi(k) = \frac{\Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p+1-b)\Gamma(a+p+1+c)}{\Gamma(k+p+1)\Gamma(a+k+p+1+c)\Gamma(p+1)\Gamma(p+1-b+c)} a_{k+p}. \]

Then, by the constraints of the hypotheses, we note that \( \Phi(k) \) is non-increasing for integers \( k \geq 1 \) and we have \( 0 < \Phi(k) < 1 \). So \( F(x) \in S_0(\alpha, \beta, \gamma, p; A) \). By Lemma 1, we get

\[ \{1+\beta[p-\gamma+\alpha(p+1)]\} \sum_{k=1}^{\infty} B_{k+p} \leq \sum_{k=1}^{\infty} \{k+\beta[p-\gamma+\alpha(k+p)]\} B_{k+p} \]

\[ \leq \sum_{k=1}^{\infty} \{k+\beta[p-\gamma+\alpha(k+p)]\} a_{k+p} \]

\[ \leq \beta(p-\gamma+\alpha p), \quad (2.6) \]

which gives

\[ \sum_{k=1}^{\infty} B_{k+p} \leq \frac{\beta(p-\gamma+\alpha p)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}}. \]

Therefore, in a similar way with the proof of [4, Theorem 2.3, p. 305], we obtain

\[ \| r_{0,A} f(A) \|_{p,q} \geq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p+1-b)\Gamma(a+p+1+c)} \| A \|^{p-b} \| A \|^{p} \]

\[ - \frac{\beta(p-\gamma+\alpha(p+1))\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}^{\gamma}(p+1-b)\Gamma(a+p+1+c)} \| A \|^{p+1-b} \| A \|^{p-b} \]

(2.7)

and

\[ \| r_{0,A} f(A) \|_{p,q} \leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p+1-b)\Gamma(a+p+1+c)} \| A \|^{p-b} \| A \|^{p} \]

\[ + \frac{\beta(p-\gamma+\alpha(p+1))\Gamma(p+1-b+c)\Gamma(p+1)}{\{1+\beta[p-\gamma+\alpha(p+1)]\}^{\gamma}(p+1-b)\Gamma(a+p+1+c)} \| A \|^{p+1-b} \| A \|^{p-b} \]

(2.8)

By equation (7) of [4, p.307],
Since $A^*A = AA^*$, $\|A\| = r_{sp}(A)$. So

$$1 = \|AA^{-1}\| \leq \|A\| \|A^{-1}\| = r_{sp}(A)r_{sp}(A^{-1}) \leq 1.$$  

Thus

$$\|A^{-1}\| = \|A\|^{-1}. \quad (2.10)$$

By (2.9) and (2.10),

$$\|A^b\| = \|A\|^b \quad (2.11)$$

for all real $b$. Therefore from (2.7), (2.8) and (2.11) we have the desired estimates.

**THEOREM 2.** Let $p > \max\{b-c-1, b, -2-c+a\}$, $c+1 < (p-b)(1-a+p+c)$, and $b(2-a+c) \leq (1-a)(1+p)$. If $f(z) \in \mathcal{S}_0(\alpha, \beta, \gamma, p; A)$, then

$$\|D_{0, A}^{\alpha,b,c} f(A)\| \leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \|A\|^{p-b-1}$$

$$+ \frac{\beta(p+1)(p-\gamma+\alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{(1+\beta[p-\gamma+\alpha(p+1)])\Gamma(p-b)\Gamma(2-a+p+c)} \|A\|^{p-b} \quad (2.12)$$

and

$$\|D_{0, A}^{\alpha,b,c} f(A)\| \geq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \|A\|^{p-b-1}$$

$$- \frac{\beta(p+1)(p-\gamma+\alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{(1+\beta[p-\gamma+\alpha(p+1)])\Gamma(p-b)\Gamma(2-a+p+c)} \|A\|^{p-b} \quad (2.13)$$

for $0 < a < 1$, $b, c \in \mathbb{R}$ and all invertible operator $A$ with $(A^\frac{1}{k})^*A^\frac{1}{k} = A^\frac{1}{k}$ $(k \in \mathbb{N})$, $\|A\| < 1$ and $r_{sp}(A)r_{sp}(A^{-1}) \leq 1$, where $r_{sp}(A)$ is the radius of spectrum of $A$.

**PROOF.** Consider the function

$$G(A) = \frac{\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(p+1-b+c)\Gamma(p+1)} A^{b+1}D_{0, A}^{\alpha,b,c} f(A)$$

$$= A^p - \sum_{k=1}^{\infty} \Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p-b)\Gamma(2-a+p+c) a_{k+p} A^{k+p}$$

$$= A^p - \sum_{k=1}^{\infty} C_{k+p} A^{k+p}, \quad (2.14)$$

where

$$C_{k+p} = \frac{\Gamma(k+p+1-b+c)\Gamma(p+1+k)\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(k+p-b)\Gamma(2-a+k+p+c)\Gamma(p+1)\Gamma(p+1-b+c)} a_{k+p}.$$ 

Hence, for convenience, we put

$$\Psi(k) = \frac{\Gamma(k+p+1-b+c)\Gamma(p+k)\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(k+p-b)\Gamma(2-a+k+p+c)\Gamma(p+1)\Gamma(p+1-b+c)} \quad (k \in \mathbb{N}) \quad (2.15)$$

Then, by the constraints of the hypotheses, we note that $\Psi(k)$ is non-increasing for integers $k \geq 1$ and we have $0 < \Psi(k) < 1$, i.e.,

$$0 < \frac{\Gamma(k+p+1-b+c)\Gamma(p+k)\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(k+p-b)\Gamma(2-a+k+p+c)\Gamma(p+1)\Gamma(p+1-b+c)} < k + p.$$ 

Also, by the relation

$$\frac{k+p}{p+1} (1+\beta[p-\gamma+\alpha(p+1)]) \leq k + \beta[p-\gamma+\alpha(p+k)] \quad (k \geq 1), \quad (2.16)$$

we get
\[
\sum_{k=1}^{\infty} \frac{k + p}{p + 1} \left\{ 1 + \beta[p - \gamma + \alpha(k + p)] \right\} \Psi(k) a_{k+p} \leq \sum_{k=1}^{\infty} \frac{k + \beta[p - \gamma + \alpha(k + p)]}{k + \beta[p - \gamma + \alpha(p + 1)]} \Psi(k) a_{k+p}
\]

that is,
\[
\sum_{k=1}^{\infty} (k + p) \Psi(k) a_{k+p} \leq \frac{\beta(p + 1)(p - \gamma + \alpha p)}{1 + \beta[p - \gamma + \alpha(p + 1)]}.
\]

Therefore, in the same way with the proof of Theorem 1, we obtain
\[
\| D_{b,c}^{p,b,c} f(A) \| \leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^{p-b-1} + \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \sum_{k=1}^{\infty} (k + p) \Psi(k) a_{k+p}
\]
\[
\leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^{p-b-1} + \frac{\beta(p+1)(p - \gamma + \alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{1 + \beta[p - \gamma + \alpha(p + 1)]} \frac{\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^{p-b}.
\] (2.18)

and
\[
\| D_{b,c}^{p,b,c} f(A) \| \geq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^{p-b-1} - \frac{\beta(p+1)(p - \gamma + \alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{1 + \beta[p - \gamma + \alpha(p + 1)]} \frac{\Gamma(p-b)\Gamma(2-a+p+c)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^{p-b}.\] (2.19)

REMARK. (i) By the proof of Theorem 1, if we put
\[
F(z) = f_{b,c}^{p,b,c}(z) = \frac{\Gamma(p+1-b+c)\Gamma(a+p+1+c)}{\Gamma(p+1-b+c)\Gamma(p+1)} z^{b} f_{b,c}^{p,b,c}(z),
\]
then we know that \( f_{b,c}^{p,b,c}(z) \) is a fractional linear operator from \( \mathbb{D}_{0}^{\alpha}, \beta, \gamma, p \) to itself.

(ii) From (1.17) it is easy to see that Theorem 1 and Theorem 2 are generalizations of [4, Theorem 3.1 and Theorem 3.2].

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