ON INVERSION OF H-TRANSFORM IN $\mathcal{L}_{\nu, r}$-SPACE

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ABSTRACT. The paper is devoted to study the inversion of the integral transform

$$(H f)(x) = \int_{0}^{\infty} H_{p,q}^{m,n} \left[ z \begin{pmatrix} a_i, & \alpha_i \end{pmatrix}_{1,p} \begin{pmatrix} b_j, & \beta_j \end{pmatrix}_{1,q} \right] f(t) dt$$

involving the $H$-function as the kernel in the space $\mathcal{L}_{\nu, r}$ of functions $f$ such that

$$\int_{0}^{\infty} |e^{\nu f(t)}| \frac{dt}{t} < \infty \quad (1 < r < \infty, \ \nu \in \mathbb{R}).$$

KEY WORDS AND PHRASES: $H$-function, Integral transform.

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1. INTRODUCTION

This paper deals with the integral transforms of the form

$$(H f)(x) = \int_{0}^{\infty} H_{p,q}^{m,n} \left[ z \begin{pmatrix} a_i, & \alpha_i \end{pmatrix}_{1,p} \begin{pmatrix} b_j, & \beta_j \end{pmatrix}_{1,q} \right] f(t) dt,$$

where $H_{p,q}^{m,n} \left[ z \begin{pmatrix} a_i, & \alpha_i \end{pmatrix}_{1,p} \begin{pmatrix} b_j, & \beta_j \end{pmatrix}_{1,q} \right]$ is the $H$-function, which is a function of general hypergeometric type being introduced by S. Pincherle in 1888 (see [2, §1.19]). For integers $m, n, p, q$ such that $0 \leq m \leq q$, $0 \leq n \leq p$, $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+ = [0, \infty)$ ($1 \leq i \leq p$, $1 \leq j \leq q$), it can be
written by
\[
\mathcal{H}^{m,n}_{p,q}\left[ z \begin{pmatrix} \alpha_1, \alpha_i \,_{1,s} \\ \beta_1, \beta_j \,_{1,q} \end{pmatrix} \right] = \mathcal{H}^{m,n}_{p,q}\left[ z \begin{pmatrix} \alpha_1, \alpha_i \\ \beta_1, \beta_j \end{pmatrix} \right]
\]
\[
= \frac{1}{2\pi i} \int_L \mathcal{G}^{m,n}_{p,q}\left[ \begin{pmatrix} \alpha_1, \alpha_i \,_{1,s} \\ \beta_1, \beta_j \,_{1,q} \end{pmatrix} \right] z^{-s} ds,
\]
where
\[
\mathcal{G}^{m,n}_{p,q}\left[ \begin{pmatrix} \alpha_1, \alpha_i \,_{1,s} \\ \beta_1, \beta_j \,_{1,q} \end{pmatrix} \right] = \prod_{j=m+1}^{p} \Gamma(\beta_j + \alpha_i s) \prod_{i=1}^{n} \Gamma(1 - \alpha_i - \alpha_i s)
\prod_{i=m+1}^{q} \Gamma(1 - \beta_j - \beta_j s),
\]
the contour \(L\) is specially chosen and an empty product, if it occurs, is taken to be one. The theory of this function may be found in Braaksma [1], Srivastava et al. [13, Chapter 1], Mathai and Saxena [8, Chapter 2] and Prudnikov et al. [9, §8.3]. We abbreviate the \(H\)-function (1.2) and the function (1.3) to \(H(z)\) and \(\mathcal{H}(s)\) when no confusion occurs. We note that the formal Mellin transform \(\mathcal{M}\) of (1.1) gives the relation
\[
(\mathcal{M}Hf)(s) = \mathcal{H}(s)(\mathcal{M}f)(1 - s).
\]
Most of the known integral transforms can be put into the form (1.1), in particular, if \(\alpha_1 = \ldots = \alpha_p = \beta_1 = \ldots = \beta_q = 1\), (1.1) is the integral transform with Meijer’s \(G\)-function in the kernel (Rooney [11], Samko et al. [12, §36]). The integral transform (1.1) with the \(H\)-function kernel or the \(H\)-transform was investigated by many authors (see Bibliography in Kilbas et al. [5-6]). In Kilbas et al. [5-7] we have studied it in the space \(L_{\nu, r}\) (\(1 \leq r < \infty\), \(\nu \in \mathbb{R}\)) consisted of Lebesgue measurable complex valued functions \(f\) for which
\[
\int_0^{\infty} \left| t^{-\nu} f(t) \right| t dt < \infty.
\]
We have investigated the mapping properties such as the boundedness, the representation and the range of the \(H\)-transform (1.1) on the space \(L_{\nu, 2}\) in Kilbas et al. [5] and on the space \(L_{\nu, r}\) with any \(1 \leq r < \infty\) in Kilbas et al. [6-7], provided that \(\delta \geq 0\), \(\delta = 1\) and \(\Delta = 0\) or \(\Delta \neq 0\), respectively. In Glaeske et al. [3] the results were extended to any \(\delta > 0\). Here
\[
a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=m+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j,
\]
\[
\delta = \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j},
\]
\[
\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i.
\]
In particular, we have proved that for certain ranges of parameters, the \(H\)-transform (1.1) have the representations
\[
(Hf)(x) = h z^{-(\lambda + 1)/h} \frac{d}{dx} z^{(\lambda + 1)/h} \int_0^{\infty} H^{m,n+1}_{p+1,q+1} \left[ \begin{pmatrix} (\lambda - h), (\alpha_i, \alpha_i) \,_{1,s} \\ (\beta_j, \beta_j) \,_{1,q} \end{pmatrix} \right] f(t) dt,
\]
or
\[
(Hf)(x) = - h z^{-(\lambda + 1)/h} \frac{d}{dx} z^{(\lambda + 1)/h} \int_0^{\infty} H^{m+1,n}_{p+1,q} \left[ \begin{pmatrix} (\alpha_i, \alpha_i) \,_{1,s} \\ (\lambda - h), (\beta_j, \beta_j) \,_{1,q} \end{pmatrix} \right] f(t) dt.
\]
owing to the value of Re(λ), where λ ∈ C and h ∈ R \ {0}.

In this paper we apply the results of Kilbas et al. [5-7] and Glaeske et al. [3] to find the inverse of the integral transforms (1.1) on the space \( L_r \), with \( 1 < r < \infty \) and \( \nu \in \mathbb{R} \). Section 2 contains preliminary information concerning the properties of the \( H \)-transform (1.1) in the space \( L_{\nu,r} \) and an asymptotic behavior of the \( H \)-function (1.2) at zero and infinity. In Sections 3 and 4 we prove that the inversion of the \( H \)-transform have the respective form (1.9) or (1.10):

\[
\int_0^\infty H_{p+1,q+1}^{r-m-p-n+1} \left[ zt \begin{array}{c}
(-\lambda, h), \left(1 - a_i - \alpha_i, \alpha_i\right)_{n+1, p}, \left(1 - a_i - \alpha_i, \alpha_i\right)_{1, n}, \\
(1 - b_j - \beta_j, \beta_j)_{m+1, q}, (1 - b_j - \beta_j, \beta_j)_{1, m}, \left(-\lambda - 1, h\right)
\end{array} \right] (Hf)(t) dt \tag{1.11}
\]
or

\[
\int_0^\infty H_{p+1,q+1}^{r-m-p-n+1} \left[ zt \begin{array}{c}
(-\lambda - 1, h), \left(1 - b_j - \beta_j, \beta_j\right)_{m+1, q}, (1 - b_j - \beta_j, \beta_j)_{1, m}, \\
(1 - a_i - \alpha_i, \alpha_i)_{n+1, p}, \left(1 - a_i - \alpha_i, \alpha_i\right)_{1, n}, \left(-\lambda - 1, h\right)
\end{array} \right] (Hf)(t) dt \tag{1.12}
\]

provided that \( a^* = 0 \). Section 3 is devoted to treat on the spaces \( L_{\nu,2} \) and \( L_{\nu,r} \) with \( \Delta = 0 \), while Section 4 on the space \( L_{\nu,r} \) with \( \Delta \neq 0 \).

The obtained results are extensions of those by Rooney [11] from \( G \)-transforms to \( H \)-transforms.

2. PRELIMINARIES

We give here some results from Kilbas et al. [5-6], Glaeske et al. [3] and from Kilbas and Saigo [4], Mathai and Saxena [8], Srivastava et al. [13] concerning the properties of \( H \)-transforms (1.1) in \( L_{\nu,r} \)-spaces and the asymptotic behavior of the \( H \)-function at zero and infinity, respectively.

For the \( H \)-function (1.2), let \( a^* \) and \( \Delta \) be defined by (1.6) and (1.8) and let

\[
\alpha = \begin{cases} 
\max \left[ -\text{Re} \left( \frac{b_1}{\beta_1} \right), \ldots, -\text{Re} \left( \frac{b_m}{\beta_m} \right) \right] & \text{if } m > 0, \\
-\infty & \text{if } m = 0;
\end{cases} \tag{2.1}
\]

\[
\beta = \begin{cases} 
\min \left[ \frac{1 - a_1}{a_1}, \ldots, \frac{1 - a_n}{a_n} \right] & \text{if } n > 0, \\
\infty & \text{if } n = 0;
\end{cases} \tag{2.2}
\]

\[
a_1^* = \sum_{j=1}^m \beta_j - \sum_{i=n+1}^p \alpha_i; \quad a_2^* = \sum_{i=1}^n \alpha_i - \sum_{j=m+1}^q \beta_j; \quad a_1^* + a_2^* = a^*; \tag{2.3}
\]

\[
\mu = \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p - q}{2}. \tag{2.4}
\]

For the function \( \mathcal{H}(s) \) given in (1.3), the exceptional set of \( \mathcal{H} \) is meant the set of real numbers \( \nu \) such that \( \alpha < 1 - \nu < \beta \) and \( \mathcal{H}(s) \) has a zero on the line \( \text{Re}(s) = 1 - \nu \) (see Rooney [11]). For two Banach space \( X \) and \( Y \) we denote by \([X, Y]\) the collection of bounded linear operators from \( X \) to \( Y \).

**THEOREM 2.1.** [5, Theorem 3], [6, Theorem 3.3] Suppose that \( \alpha < 1 - \nu < \beta \) and that either \( a^* > 0 \) or \( a^* = 0 \), \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 0 \). Then

(a) There is a one-to-one transform \( H \in [L_{\nu,2}, L_{1-\nu,2}] \) so that (1.4) holds for \( f \in L_{\nu,2} \) and \( \text{Re}(s) = 1 - \nu \). If \( a^* = 0 \), \( \Delta(1 - \nu) + \text{Re}(\mu) = 0 \) and \( \nu \) is not in the exceptional set of \( \mathcal{H} \), then the operator \( H \) transforms \( L_{\nu,2} \) onto \( L_{1-\nu,2} \).
(b) If \( f \in \mathcal{L}_{\nu,2} \) and \( \text{Re}(\lambda) > (1-\nu)h - 1 \), \( Hf \) is given by (1.9). If \( f \in \mathcal{L}_{\nu,2} \) and \( \text{Re}(\lambda) < (1-\nu)h - 1 \), then \( Hf \) is given by (1.10).

**Theorem 2.2.** [6, Theorem 4.1], [3, Theorem 1] Let \( a^* = \Delta = 0, \text{Re}(\mu) = 0 \) and \( \alpha < 1 - \nu < \beta \).

(a) The transform \( H \) is defined on \( \mathcal{L}_{\nu,2} \) and it can be extended to \( \mathcal{L}_{\nu,r} \) as an element of \( [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-v,r}] \) for \( 1 < r < \infty \).

(b) If \( 1 < r \leq 2 \), the transform \( H \) is one-to-one on \( \mathcal{L}_{\nu,r} \) and there holds the equality

\[
(\mathcal{M}Hf)(s) = \mathcal{K}(s)(\mathcal{M}f)(1-s), \quad \text{Re}(s) > 0.
\]

(2.5)

(c) If \( f \in \mathcal{L}_{\nu,r} \) (\( 1 < r < \infty \)), then \( Hf \) is given by (1.9) for \( \text{Re}(\lambda) > (1-\nu)h - 1 \), while \( Hf \) is given by (1.10) for \( \text{Re}(\lambda) < (1-\nu)h - 1 \).

**Theorem 2.3.** [6, Theorem 5.1], [3, Theorem 3] Let \( a^* = 0, \Delta > 0, \alpha < 1 - \nu < \beta, 1 < r < \infty \) and \( \Delta(1-\nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \), where

\[
\gamma(r) = \max \left[ \frac{1}{r}, \frac{1}{r'} \right] \quad \text{with} \quad \frac{1}{r} + \frac{1}{r'} = 1.
\]

(2.6)

(a) The transform \( H \) is defined on \( \mathcal{L}_{\nu,2} \), and it can be extended to \( \mathcal{L}_{\nu,r} \) as an element of \( [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-v,r}] \) for all \( s \) with \( r \leq s < \infty \) such that \( s' \geq [1/2 - \Delta(1-\nu) - \text{Re}(\mu)]^{-1} \) with \( 1/s + 1/s' = 1 \).

(b) If \( 1 < r \leq 2 \), the transform \( H \) is one-to-one on \( \mathcal{L}_{\nu,r} \) and there holds the equality (2.5).

(c) If \( f \in \mathcal{L}_{\nu,r} \) and \( g \in \mathcal{L}_{\nu,s} \) with \( 1 < r < \infty, 1 < s < \infty, 1/r + 1/s \geq 1 \) and \( \Delta(1-\nu) + \text{Re}(\mu) \leq 1/2 - \max[\gamma(r), \gamma(s)] \), then the relation

\[
\int_0^\infty f(x)(Hg)(x)dx = \int_0^\infty g(x)(Hf)(x)dx
\]

(2.7)

holds.

The following two assertions give the asymptotic behavior of the the \( H \)-function (1.2) at zero and infinity provided that the poles of Gamma functions in the numerator of \( \mathcal{F}(s) \) do not coincide, i.e.

\[
\beta_j(a_i - 1 - k) \neq \alpha_i(b_j + l) \quad (i = 1, \ldots, n; j = 1, \ldots, m; k, l = 0, 1, 2, \ldots).
\]

(2.8)

**Theorem 2.4.** [8, §1.1.6], [13, §2.2] Let the condition (2.8) be satisfied and poles of Gamma functions \( \Gamma(b_j + \beta_j s) \) \((j = 1, \ldots, m)\) be simple, i.e.

\[
\beta_i(b_i + k) \neq \beta_j(b_j + l) \quad (i \neq j; i, j = 1, \ldots, m; k, l = 0, 1, 2, \ldots).
\]

(2.9)

If \( \Delta \geq 0 \), then

\[
H_{p,q}^{m,n}(x) = O(x^\rho) \quad (|x| \to 0) \quad \text{with} \quad \rho = \min_{1 \leq i, j \leq m} \left[ \frac{\text{Re}(b_j)}{\beta_j} \right].
\]

(2.10)

**Theorem 2.5.** [4, Corollary 3] Let \( a^*, \Delta \) and \( \mu \) be given by (1.6), (1.8) and (2.4), respectively. Let the conditions in (2.8) be satisfied and poles of Gamma functions \( \Gamma(1-a_i - \alpha_i s) \) \((i = 1, \ldots, n)\) be simple, i.e.

\[
\alpha_j(1-a_i + k) \neq \alpha_i(1-a_j + l) \quad (i \neq j; i, j = 1, \ldots, n; k, l = 0, 1, 2, \ldots).
\]

(2.11)
If \( a^* = 0 \) and \( \Delta > 0 \), then

\[
H_{p,a}^n(z) = O(z^\varphi) \quad (|z| \to \infty) \quad \text{with} \quad \varphi = \max \left[ \max_{1 \leq \alpha \leq n} \left( \frac{\text{Re}(\alpha)}{\alpha} - 1 \right), \frac{\text{Re}(\mu) + 1/2}{\Delta} \right]. \quad (2.12)
\]

**REMARK 2.1.** It was proved in Kilbas and Saigo [4, §6] that if poles of Gamma functions \( \Gamma(1 - \alpha_i - \alpha_i s) \) \((i = 1, \cdots, n)\) are not simple (i.e. conditions in (2.11) are not satisfied), then the \( H \)-function (1.1) have power-logarithmic asymptotics at infinity. In this case the logarithmic multiplier \( \log(z)^N \) with \( N \) being the maximal number of orders of the poles may be added to the power multiplier \( z^\varphi \) and hence the asymptotic estimate \( O(z^\varphi) \) in (2.12) may be replaced by \( O(z^{\varphi \log(z)^N}) \). The same result is valid in the case of the asymptotics of the \( H \)-function (1.1) at zero, and the estimate \( O(z^\varphi) \) in (2.10) may be replaced by \( O(z^{\varphi \log(z)^M}) \), where \( M \) is the maximal number of orders of the points at which the poles of \( \Gamma(b_j + \beta_j s) \) \((j = 1, \cdots, m)\) coincide.

### 3. INVERSION OF \( H \)-TRANSFORM IN \( \mathcal{L}_{v,2} \) AND \( \mathcal{L}_{v,r} \) WHEN \( \Delta = 0 \)

In this and next sections we investigate that \( H \)-transform will have the inverse of the form (1.11) or (1.12). If \( f \in \mathcal{L}_{v,2} \) and \( H \) is defined on \( \mathcal{L}_{v,r} \), then according to Theorem 2.2, the equality (2.5) holds under the assumption there. This fact implies the relation

\[
(\mathcal{M}f)(s) = \frac{(\mathcal{M}Hf)(1 - s)}{\mathcal{H}(1 - s)} \quad (\text{Re}(s) = \nu). \quad (3.1)
\]

for \( \text{Re}(s) = \nu \). By (1.3) we have

\[
\frac{1}{\mathcal{H}(1 - s)} = \mathcal{H}_{-m,n}^{p,q} \left[ \begin{array}{c} (1 - a_i - \alpha_i, \alpha_i)_{n+1,m+1}, (1 - a_i - \alpha_i, \alpha_i)_{1,n} \\ (1 - b_j - \beta_j, \beta_j)_{m+1,n}, (1 - b_j - \beta_j, \beta_j)_{1,m} \end{array} \right] s \equiv \mathcal{K}_0(s), \quad (3.2)
\]

and hence (3.1) takes the form

\[
(\mathcal{M}f)(s) = (\mathcal{M}Hf)(1 - s)\mathcal{K}_0(s) \quad (\text{Re}(s) = \nu). \quad (3.3)
\]

We denote by \( \alpha_0, \beta_0, \alpha_0^*, \alpha_0^*, \alpha_0^*, \delta_0, \Delta_0 \) and \( \mu_0 \) for \( \mathcal{K}_0 \) instead of those for \( \mathcal{H} \). Then we find

\[
\alpha_0 = \max \left\{ \begin{array}{ll} \frac{\text{Re}(b_{m+1}) - 1}{\beta_{m+1}} + 1, \cdots, \frac{\text{Re}(b_q) - 1}{\beta_q} + 1 \quad & \text{if} \quad q > m, \\ -\infty \quad & \text{if} \quad q = m; \end{array} \right. \quad (3.4)
\]

\[
\beta_0 = \min \left\{ \begin{array}{ll} \frac{\text{Re}(a_{n+1}) + 1}{\alpha_{n+1}} + 1, \cdots, \frac{\text{Re}(a_p) + 1}{\alpha_p} + 1 \quad & \text{if} \quad p > n, \\ \infty \quad & \text{if} \quad p = n; \end{array} \right. \quad (3.5)
\]

\[
a_0^* = -a_0^*; \quad a_0^* = -a_0^*; \quad a_0^* = -a_0^*; \quad \delta_0 = \delta; \quad \Delta_0 = \Delta; \quad \mu_0 = -\mu - \Delta. \quad (3.6)
\]

We also note that if \( \alpha_0 < \nu < \beta_0, \nu \) is not in the exceptional set of \( \mathcal{K}_0 \).

First we consider the case \( r = 2 \).

**THEOREM 3.1.** Let \( \alpha < 1 - \nu < \beta, \alpha_0 < \nu < \beta_0, a^* = 0 \) and \( \Delta(1 - \nu) + \text{Re}(\mu) = 0 \). If \( f \in \mathcal{L}_{v,2} \), the relation (1.11) holds for \( \text{Re}(\lambda) > \nu h - 1 \) and the relation (1.12) holds for \( \text{Re}(\lambda) < \nu h - 1 \).
PROOF. We apply Theorem 2.1 with \( \mathcal{K} \) being replaced by \( \mathcal{K}_0 \) and \( \nu \) by \( 1 - \nu \). By the assumption and (3.6) we have

\[
\alpha_0 = -a^* = 0, \quad (3.7)
\]

\[
\Delta_0[1 - (1 - \nu)] + \operatorname{Re}(\mu_0) = \Delta \nu - \operatorname{Re}(\mu) - \Delta = -[\Delta(1 - \nu) + \operatorname{Re}(\mu)] = 0 \quad (3.8)
\]

and \( \alpha_0 < 1 - (1 - \nu) < \beta_0 \), and thus Theorem 2.1(a) applies. Then there is a one-to-one transform \( H_0 \in [\mathcal{L}_{1-\nu}, \mathcal{L}_r] \) so that the relation

\[
(\mathfrak{M}H_0f)(s) = \mathcal{K}_0(s)(\mathfrak{M}f)(1 - s)
\]

holds for \( f \in \mathcal{L}_{1-\nu} \) and \( \operatorname{Re}(s) = \nu \). Further if \( f \in \mathcal{L}_r \), \( Hf \in \mathcal{L}_{1-\nu} \) and it follows from (3.9), (1.4) and (3.2) that

\[
(\mathfrak{M}H_0Hf)(s) = \mathcal{K}_0(s)(\mathfrak{M}f)(1 - s) = \mathcal{K}_0(s)\mathcal{K}((1 - s)(\mathfrak{M}f)(s) = (\mathfrak{M}f)(s),
\]

if \( \operatorname{Re}(s) = \nu \). Hence \( \mathfrak{M}H_0Hf = \mathfrak{M}f \) and

\[
H_0Hf = f \quad \text{for} \quad f \in \mathcal{L}_r.
\]

Applying Theorem 2.1(b) with \( \mathcal{K} \) being replaced by \( \mathcal{K}_0 \) and \( \nu \) by \( 1 - \nu \), we obtain for \( f \in \mathcal{L}_{1-\nu} \) that

\[
(H_0f)(x) = hx^{1-(\lambda+1)/h} d^2 x^{(\lambda+1)/h}
\]

\[
- \int_0^\infty \mathcal{H}_2^{\frac{\lambda-n-m}{4}+1} \left[ \begin{array}{c}
(-\lambda, h), (1 - a_i - \alpha_i, \alpha_i)_{n+1}, (1 - a_i, -\alpha_i, \alpha_i)_{1,n} \\
(1 - b_i - \beta_i, \beta_i)_{m+1}, (1 - b_i - \beta_i, \beta_i)_{1,m}, (\lambda - 1 - h) 
\end{array} \right] f(t) dt, (3.11)
\]

if \( \operatorname{Re}(\lambda) > [1 - (1 - \nu)]h - 1 \) and

\[
(H_0f)(x) = -hx^{1-(\lambda+1)/h} d^2 x^{(\lambda+1)/h}
\]

\[
- \int_0^\infty \mathcal{H}_2^{\frac{\lambda-n-m}{4}+1} \left[ \begin{array}{c}
(-a_i - \alpha_i, \alpha_i)_{n+1}, (1 - a_i - \alpha_i, \alpha_i)_{1,n}, (\lambda - h) \\
(\lambda - 1 - h), (1 - b_i - \beta_i, \beta_i)_{m+1}, (1 - b_i - \beta_i, \beta_i)_{1,m} 
\end{array} \right] f(t) dt, (3.12)
\]

if \( \operatorname{Re}(\lambda) < [1 - (1 - \nu)]h - 1 \). Replacing \( f \) by \( Hf \) and using (3.10) we have the relations (1.11) and (1.12) for \( f \in \mathcal{L}_r \), if \( \operatorname{Re}(\lambda) > \nu h - 1 \) and \( \operatorname{Re}(\lambda) < \nu h - 1 \), respectively, which completes the proof of theorem.

Next results is the extension of Theorem 3.1 to \( \mathcal{L}_{r,r} \)-spaces for any \( 1 < r < \infty \), provided that \( \Delta = 0 \) and \( \operatorname{Re}(\mu) = 0 \).

THEOREM 3.2. Let \( \alpha < 1 - \nu < \beta, \alpha_0 < \nu < \beta_0, a^* = 0, \Delta = 0 \) and \( \operatorname{Re}(\mu) = 0 \). If \( f \in \mathcal{L}_{r,r} \) \( (1 < r < \infty) \), the relation (1.11) holds for \( \operatorname{Re}(\lambda) > \nu h - 1 \) and the relation (1.12) holds for \( \operatorname{Re}(\lambda) < \nu h - 1 \).

PROOF. We apply Theorem 2.2 with \( \mathcal{K} \) being replaced by \( \mathcal{K}_0 \) and \( \nu \) by \( \nu - 1 \). By the assumption and (3.6), we have \( a_0^* = \Delta_0 = 0, \operatorname{Re}(\mu_0) = 0 \) and \( \alpha_0 < 1 - (1 - \nu) < \beta_0 \), and thus Theorem 2.2(a) can be applied. In accordance with this theorem, \( H_0 \) can be extended to \( \mathcal{L}_{1-\nu,r} \) as an element of \( H_0 \in [\mathcal{L}_{1-\nu,r}, \mathcal{L}_r] \). By virtue of (3.10) \( H_0H \) is identical operator in \( \mathcal{L}_{r,2} \). By Rooney [11, Lemma 2.2] \( \mathcal{L}_{r,2} \) is dense in \( \mathcal{L}_{r,r} \) and since \( H \in [\mathcal{L}_{r,r}, \mathcal{L}_{1-\nu,r}] \) and \( H_0 \in [\mathcal{L}_{1-\nu,r}, \mathcal{L}_{r,r}] \), the operator \( H_0H \) is identical in \( \mathcal{L}_{r,r} \) and hence

\[
H_0Hf = f \quad \text{for} \quad f \in \mathcal{L}_{r,r}.
\]
Applying Theorem 2.2(c) with $K$ being replaced by $K_0$ and $\nu$ by $1 - \nu$, we obtain that the relations (3.11) and (3.12) hold for $f \in \mathcal{L}_{1-r}$, when $\text{Re}(\lambda) > [1 - (1 - \nu)]h - 1$ and $\text{Re}(\lambda) < [1 - (1 - \nu)]h - 1$, respectively. Replacing $f$ by $Hf$ and using (3.13), we arrive at (1.11) and (1.12) for $f \in \mathcal{L}_{1-r}$, if $\text{Re}(\lambda) > \nu h - 1$ and $\text{Re}(\lambda) < \nu h - 1$, respectively, which completes the proof of theorem.

**REMARK 3.1.** If $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$ which means that the $H$-function (1.2) is Meijer's $G$-function, then $\Delta = q - p$ and Theorems 8.1 and 8.2 in Rooney [11] follow from Theorems 3.1 and 3.2.

### 4. INVERSION OF $H$-TRANSFORM IN $\mathcal{L}_{1-r}$ WHEN $\Delta \neq 0$

We now investigate under what condition the $H$-transform with $\Delta \neq 0$ will have the inverse of the form (1.11) or (1.12). First, we consider the case $\Delta > 0$. To obtain the inversion of the $H$-transform on $\mathcal{L}_{1-r}$ we use the relation (2.7).

**THEOREM 4.1.** Let $0 < r < \infty, -\infty < \alpha < 1 - \nu < \beta, \alpha_0 < \nu < \min\{\beta_0, [\text{Re}(\mu + 1/2)/\Delta] + 1\}, a^* = 0, \Delta > 0$ and $\Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r)$, where $\gamma(r)$ is given in (2.6). If $f \in \mathcal{L}_{1-r}$, then the relations (1.11) and (1.12) hold for $\text{Re}(\lambda) > \nu h - 1$ and for $\text{Re}(\lambda) < \nu h - 1$, respectively.

**PROOF.** According to Theorem 2.3(a), the $H$-transform is defined on $\mathcal{L}_{1-r}$. First we consider the case $\text{Re}(\lambda) > \nu h - 1$. Let $H_1(t)$ be the function

$$H_1(t) = H^{r-\alpha_0-\beta_0+1}_{r+\alpha_0+1}(t)$$

We prove that $H_1(t)$ for any $s (1 \leq s < \infty)$. For this, we first apply Theorems 2.4 and 2.5 and Remark 2.1 to $H_1(t)$ to find its asymptotic behavior at zero and infinity. According to (3.4), (3.5) and the assumptions, we find

$$\frac{\text{Re}(b_j) - 1}{\beta_j} + 1 \leq \alpha_0 < \beta_0 \leq \frac{\text{Re}(a_i)}{\alpha_i} + 1 \quad (j = m + 1, \cdots, q; i = n + 1, \cdots, p);$$

$$\frac{\text{Re}(b_j) - 1}{\beta_j} + 1 \leq \alpha_0 < \nu < \frac{\text{Re}(\lambda) + 1}{h} \quad (j = m + 1, \cdots, q).$$

Then it follows from here that the poles

$$a_{ik} = \frac{a_i + k}{a_i} + 1 \quad (i = n + 1, \cdots, p; k = 0, 1, 2, \cdots), \quad \lambda_n = \frac{\lambda + 1 + n}{h} \quad (n = 0, 1, 2, \cdots)$$

of Gamma functions $\Gamma(a_i + \alpha_i - \alpha_i s) (i = n + 1, \cdots, p)$ and $\Gamma(1 + \lambda - h s)$, and the poles

$$b_{ij} = \frac{b_j - 1 - l}{\beta_j} + 1 \quad (j = m + 1, \cdots, q; l = 0, 1, 2, \cdots)$$

of Gamma functions $\Gamma(1 - b_j - \beta_j + \beta_j s) (j = m + 1, \cdots, q)$ do not coincide. Hence by Theorem 2.4, (4.1) and Remark 2.1, we have

$$H_1(t) = O(t^{\rho_1}) \quad (|t| \to 0)$$

with

$$\rho_1 = \min_{m+1 \leq s \leq q} \left\{ \frac{1 - \text{Re}(b_j)}{\beta_j} \right\} - 1 = -\alpha_0.$$
for $\alpha_0$ being given in (3.4), or

$$H_1(t) = O(t^{-\alpha_0}) \quad (t \to 0)$$

(4.3)

with an additional logarithmic multiplier $[\log t]^N$ possibly, if Gamma functions $\Gamma(1 - b_j - \beta_j + \beta_j s)$ ($j = m + 1, \cdots, q$) have general poles of order $N \geq 2$ at some points.

Further by Theorem 2.5, (4.1) and Remark 2.1,

$$H_1(t) = O(t^{\gamma_1}) \quad (t \to \infty) \quad \text{with} \quad \gamma_1 = \max \left[ \beta_0, \frac{-\text{Re}(\mu) - 1/2}{\Delta} - 1, \frac{-\text{Re}(\lambda) - 1}{h} \right]$$

(4.4)

for $\beta_0$ being given by (3.5), or

$$H_1(t) = O(t^{-\gamma_0}) \quad (|t| \to \infty) \quad \text{with} \quad \gamma_0 = \min \left[ \beta_0, \frac{\text{Re}(\mu) + 1/2}{\Delta} + 1, \frac{\text{Re}(\lambda) + 1}{h} \right]$$

(4.5)

and with an additional logarithmic multiplier $[\log t]^M$ possibly, if Gamma functions $\Gamma(1 + \lambda - hs)$, $\Gamma(\alpha_i + \alpha_i - \alpha_i s)$ ($i = n + 1, \cdots, p$) have general poles of order $M \geq 2$ at some points.

Let Gamma functions $\Gamma(1 - b_j - \beta_j + \beta_j s)$ ($j = m + 1, \cdots, q$) or $\Gamma(1 + \lambda - hs)$, $\Gamma(\alpha_i + \alpha_i - \alpha_i s)$ ($i = n + 1, \cdots, p$) have simple poles. Then from (4.3) and (4.4) we see that for $1 \leq s < \infty$, $H_1(t) \in \mathcal{L}_{s'}$, if and only if, for some $R_1$ and $R_2$, $0 < R_1 < R_2 < \infty$, the integrals

$$\int_0^{R_1} t^{(\nu-\alpha_0)-1} dt, \quad \int_{R_2}^{\infty} t^{(\nu-\gamma_0)-1} dt$$

(4.6)

are convergent. Since by the assumption $\nu > \alpha_0$, the first integral in (4.6) converges. In view of our assumptions

$$\nu < \beta_0, \quad \nu < \frac{\text{Re}(\mu) + 1/2}{\Delta} + 1, \quad \nu < \frac{\text{Re}(\lambda) + 1}{h}$$

we find $\nu - \gamma_0 < 0$ and the second integral in (4.6) converges, too.

If Gamma functions $\Gamma(1 - b_j - \beta_j + \beta_j s)$ ($j = m + 1, \cdots, q$) or $\Gamma(1 + \lambda - hs)$, $\Gamma(\alpha_i + \alpha_i - \alpha_i s)$ ($i = n + 1, \cdots, p$) have general poles, then the logarithmic multipliers $[\log t]^N$ ($N = 1, 2, \cdots$) may be added in the integrals in (4.6), but they do not influence on the convergence of them. Hence, under the assumptions we have

$$H_1(t) \in \mathcal{L}_{s'} \quad (1 \leq s < \infty).$$

(4.7)

Let $a$ be a positive number and $\Pi_a$ denote the operator

$$(\Pi_a f)(x) = f(ax) \quad (x > 0)$$

(4.8)

for a function $f$ defined almost everywhere on $(0, \infty)$. It is known in Rooney [11, p.268] that $\Pi_a$ is a bounded isomorphism of $\mathcal{L}_{s'}$ onto $\mathcal{L}_{as'}$, and if $f \in \mathcal{L}_{s'}$ ($1 \leq r \leq 2$), there holds the relation for the Mellin transform $\mathfrak{M}$

$$(\mathfrak{M} \Pi_a f)(s) = a^{-s} (\mathfrak{M} f) \left( \frac{s}{a} \right) (\text{Re}(s) = \nu).$$

(4.9)

By virtue of Theorem 2.3(c) and (4.6), if $f \in \mathcal{L}_{s'}$ and $H_1 \in \mathcal{L}_{as'}$ (and hence $\Pi_a H_1 \in \mathcal{L}_{as'}$), then

$$\int_0^\infty H_1(t) (H f)(t) dt = \int_0^\infty (\Pi_a H_1)(t) (H f)(t) dt = \int_0^\infty (H \Pi_a H_1)(t) f(t) dt.$$  

(4.10)

(4.11)

From the assumption $\Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \leq 0$, Theorem 2.3(b) and (4.8) imply that

$$\mathfrak{M} H_1(s) = \mathcal{H}(s) (\mathfrak{M} \Pi_a H_1)(1 - s) = x^{-(1-s)}(\mathfrak{M} H_1)(1-s).$$

(4.12)
for Re(s) = 1 − ν. Now from (4.6), \(H_1(t) \in \mathcal{L}_{ν,1}\). Then by the definitions of the \(H\)-function (1.2), (1.3) and the direct and inverse Mellin transforms (see, for example, Samko et al. [12, (1.112), (1.113)]), we have

\[
(\mathcal{M}H_1)(s) = \mathcal{M}^{p+1,q+1}_{p+1,q+1} \left[ \begin{array}{c} (-\lambda, h), (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{1,n} \\ (1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m}, (-\lambda - 1, h) \end{array} \right] s^{-1} \\
= \mathcal{M}^{p-m,q-n}_{p,q} \left[ \begin{array}{c} (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{1,n} \\ (1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m} \end{array} \right] \frac{\Gamma(1 + \lambda - hs)}{\Gamma(2 + \lambda - hs)}
\]

for Re(s) = ν, where \(\mathcal{H}_0\) is given by (3.2). It follows from here that for Re(s) = 1 − ν,

\[
(\mathcal{M}H_1)(1 - s) = \frac{\mathcal{H}_0(1 - s)}{1 + \lambda - h(1 - s)} = \frac{1}{\mathcal{H}(s)[1 + \lambda - h(1 - s)]}.
\]

Substituting this into (4.10) we obtain

\[
(\mathcal{M}H_1)(1 - s) = \frac{x^{-(1-s)}}{1 + \lambda - h(1 - s)} \quad (\text{Re}(s) = 1 - \nu).
\]

For \(x > 0\) let us denote by \(g_x(t)\) a function

\[
g_x(t) = \begin{cases} \frac{1}{h} t^{(\lambda+1)/h - 1} & \text{if } 0 < t < x, \\
0 & \text{if } t > x,
\end{cases}
\]

then

\[
(\mathcal{M}g_x)(s) = \frac{x^{(\lambda+1)/h - 1}}{1 + \lambda - h(1 - s)},
\]

and (4.11) takes the form

\[
(\mathcal{M}H_1)(s) = (\mathcal{M}[x^{-(\lambda+1)/h} g_x])(s),
\]

which implies

\[
(H_1)(s) = x^{-(\lambda+1)/h} g_x(t).
\]

Substituting (4.13) into (4.9), we have

\[
\int_0^x H_1(xt)(Hf)(t)dt = x^{-(\lambda+1)/h} \int_0^x g_x(t)f(t)dt
\]
or, in accordance with (4.12),

\[
\int_0^x t^{(\lambda+1)/h - 1} f(t)dt = h x^{(\lambda+1)/h} \int_0^x H_1(xt)(Hf)(t)dt.
\]

Differentiating this relation we obtain

\[
f(x) = h x^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^x H_1(xt)(Hf)(t)dt
\]

which shows (1.11).

If Re(\(\lambda\)) < \(\nu h - 1\), the relation (1.12) is proved similarly to (1.11), by taking the function

\[
H_2(t) = H^{p-m+1,p-n}_{p+1,q+1} \left[ \begin{array}{c} (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{1,n}, (-\lambda, h) \\ (-\lambda - 1, h), (1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m} \end{array} \right]
\]
instead of the function $H_1(t)$ in (4.1). This completes the proof of the theorem.

In the case $\Delta < 0$ the following statement gives the inversion of $H$-transform on $\mathcal{L}_{a,r}$.

**Theorem 4.2.** Let $1 < r < \infty, \alpha < 1 - \nu < \beta < \infty, \max\{\alpha_0, \{\Re(\mu+1/2)/\Delta\}+1\} < \nu < \beta_0, \alpha > 0, \Delta < 0$ and $\Delta(1 - \nu) + \Re(\mu) \leq 1/2 - \gamma(r)$, where $\gamma(r)$ is given by (2.6). If $f \in \mathcal{L}_{a,r}$, then the relations (1.11) and (1.12) holds for $\Re(\lambda) > \nu h - 1$ and for $\Re(\lambda) < \nu h - 1$, respectively.

This theorem is proved similarly to Theorem 4.1, if we apply Theorem 5.2 from Kilbas et al. [6] instead of Theorem 2.3 and take into account the asymptotics of the $H$-function at zero and infinity (see Srivastava et al. [13, §2.2] and Kilbas and Saigo [4, Corollary 4]).

**Remark 4.1.** If $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$, then Theorems 8.3 and 8.4 in Rooney [11] follow from Theorems 4.1 and 4.2.

**References**


