ON COUNTABLE CONNECTED HAUSDORFF SPACES IN WHICH
THE INTERSECTION OF EVERY PAIR OF CONNECTED
SUBSETS IS CONNECTED

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ABSTRACT. We prove that a countable connected Hausdorff space in which the intersection of
every pair of connected subsets is connected, cannot be locally connected, and also that every
continuous function from a countable connected, locally connected Hausdorff space, to a countable
connected Hausdorff space in which the intersection of every pair of connected subsets is connected,
is constant.

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1. INTRODUCTION.

The problem of existence of countable connected Hausdorff space in which the intersection of
every pair of connected subsets is connected was posed by Ćwik in [1], and was answered in [2].
Recently, Gruenhage [3] assuming the continuum hypothesis constructed a perfectly normal space
in which the only non-degenerate connected subsets of it, are the cofinite sets. Also assuming
Martin's Axiom he constructed a completely regular and a countable Hausdorff space with this
property. Obviously, in these spaces the intersection of every pair of connected subsets is connected.
None of the spaces in [2] and [3] is locally connected, or has a dispersion point.

We prove that a countable connected Hausdorff space in which the intersection of every pair
of connected subsets is connected, cannot be locally connected, and also that every continuous
function from a countable connected, locally connected Hausdorff space, to a countable connected
Hausdorff space in which the intersection of every pair of connected subsets is connected, is con-
stant. Both these results hold in a Hausdorff connected space with a dispersion point: The first is obvious and the second, for not necessarily countable spaces, was proved by Coppin in [4]. Improvements of Coppin's result, as well as results concerning the constancy of functions between two spaces, can be found in the papers by Chew and Doyle [5], and by Sanderson [6].

Let $X$ be a connected topological space. A point $t$ is called a cut point of $X$ if the space $X \setminus \{t\}$ is not connected. Thus, it $t$ is a cut point of $X$, then the subspace $X \setminus \{t\}$ is the union of two mutually separated sets $A(t), B(t)$. (Two sets $A, B$ are called separated if $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$.) Obviously, if $A(t), B(t)$ are connected, the separation is unique. Let $x, y \in X$. A cut point $t$ of $X$ is said to separate the points $x, y$ if the above sets $A(t), B(t)$ can be chosen so that $x \in A(t)$ and $y \in B(t)$. The set of cut points of $X$ separating the points $x, y$ will be denoted by $E(x, y)$. The empty set and the singletons are considered to be connected. All spaces are assumed to have more than one point.

2. RESULTS.

PROPOSITION 1. Let $X$ be a Hausdorff connected space such that $E(a, b) \neq \emptyset$, for every $a, b \in X$. Then there exists a continuous non-constant real valued function on $X$, separating the points $a$ and $b$.

PROOF. The proof is reduced to the Urysohn's Lemma in the following manner: For every point $t \in E(a, b)$ there exist two sets $M_a(t)$, $M_b(t)$ such that $a \in M_a(t)$, $b \in M_b(t)$, $\overline{M_a(t)} = M_a(t) \cup \{t\}$, $\overline{M_b(t)} = M_b(t) \cup \{t\}$ and $X \setminus \{t\} = M_a(t) \cup M_b(t)$. Hence the sets

$$F_1 = \bigcup_{t \in E(a, b)} (M_a(t) \cup \{t\})$$

and

$$F_2 = \bigcap_{t \in E(a, b)} (M_a(t) \cup \{t\})$$

are both closed disjoint containing the points $a, b$ respectively, and not containing any cut point of $X$ separating the points $a, b$. Consequently, for every point $d$ of the set of positive dyadic rational numbers we can define an open set $(M_a(t))(d)$ such that if $d < r$, then $\overline{M_a(t)(d)} \subseteq M_a(t)(r)$. But then the function $f(x) = \inf \{d : x \in (M_a(t))(d)\}$, if $x \notin F_2$, and $f(x) = 1$, if $x \in F_2$ is continuous separating the points $a, b$.

PROPOSITION 2. Does not exist a countable connected, locally connected Hausdorff space in which the intersection of every pair of connected subsets is connected.

PROOF. As it is proved in [7, Theorem 9.1] a connected locally connected space $X$ is a Hausdorff space in which the intersection of every pair (indeed every collection) of connected sets is connected, if and only if no two point of $X$ are conjugate. That is, $E(x, y) \neq \emptyset$, for every $x, y \in X$. But then, Proposition 1 implies that there exists a non-constant continuous real valued function on $X$, which is impossible for countable connected spaces.

PROPOSITION 3. Let $X$ be a countable connected Hausdorff space in which the intersection of every pair of connected subsets is connected. Then

1. The subset $D$ of $X$ at every point of which $X$ is not locally connected, is dense.
(2) The subset \( L \) at every point of which \( X \) is locally connected is totally disconnected or empty.

PROOF (1). By Proposition 2, \( D \neq \emptyset \). Hence at every point \( x \in X \setminus \overline{D} \), the space \( X \) is locally connected and therefore if \( U_x \) is an open connected neighbourhood of \( x \) for which \( U_x \cap \overline{D} \neq \emptyset \), then \( U_x \) is also a locally connected space in which the intersection of every pair of connected subsets is connected, which is impossible, by Proposition 2.

(2). Obvious.

THEOREM. Every continuous function from a countable connected, locally connected Hausdorff space, to a connected Hausdorff space in which the intersection of every pair of connected subsets is connected, is constant.

PROOF. Let \( f \) be a continuous non-constant function from \( X \) to \( Y \). Obviously the space \( Z = f(X) \) is countable connected Hausdorff in which the intersection of every pair of connected subsets is connected. Let \( x, y \) be distinct points of \( X \) such that \( f(x) \neq f(y) \) and let \( U_{f(x)}, U_{f(y)} \) be disjoint open neighbourhoods of \( f(x), f(y) \), respectively. Since \( X \) is locally connected there exists an open connected neighbourhood \( U_x \) of \( x \) such that \( f(U_x) \subseteq U_{f(x)} \). If \( f(U_x) = \{f(z)\} \) then we consider the set \( A = \{a \in X : f(a) = f(x)\} \). Since the set \( \overline{A} \setminus \overline{A} \) is not empty, it follows that there exist a point \( a \in \overline{A} \setminus \overline{A} \) and a connected open neighbourhood \( U_a \) of \( a \), such that \( f(U_a) \subseteq U_{f(x)} \) and \( f(U_a) \neq \{f(z)\} \). Therefore the component \( C_{f(x)} \) of \( f(x) \) in \( \overline{U_{f(x)}} \) is not a singleton.

Consider the component \( K \) of \( f(y) \) in \( Z \setminus C_{f(x)} \). If \( K = \{f(y)\} \) then for the component \( M \) of \( Y \) in \( X \setminus f^{-1}(C_{f(x)}) \) it holds that \( f(M) = \{f(y)\} \) and \( f(M) = \{f(y)\} \). Since the subspace \( X \setminus f^{-1}(C_{f(x)}) \) is locally connected it follows that \( M \) is open-and-closed (in \( X \setminus f^{-1}(C_{f(x)}) \)) and hence \( M \cap f^{-1}(C_{f(x)}) \neq \emptyset \) which is impossible. Therefore the component \( K \) of \( y \) in \( Z \setminus C_{f(x)} \) is not a singleton.

Thus, by [8, Vol. II, Ch. V, Theorem 5, III], for the connected subsets \( C_{f(x)} \) and \( K \) it follows that the set \( Z \setminus K \) is connected and hence either (1) \((Z \setminus K) \cap K \neq \emptyset \), or (2) \((Z \setminus K) \cap K \neq \emptyset \) or (3) \((Z \setminus K) \cap K \neq \emptyset \).

In case (1), let \( p, q \in (Z \setminus K) \cap K \), and \( p \neq q \). Then for the connected subsets \((Z \setminus K) \cup \{p, q\}\) and \( K \) it holds that \(((Z \setminus K) \cup \{p, q\}) \cap K = \{p, q\}\), which is impossible because by assumption the intersection of every pair of connected subsets of \( Z \) must be connected. Therefore \((Z \setminus K) \cap K \) is a singleton. We set \((Z \setminus K) \cap K = \{p\}\). The set \( K \) is closed because if \( a \) is a limit point of \( K \) and \( a \notin K \) then for the connected subsets \( K \cup \{a\} \) and \((Z \setminus K) \) the subset \((Z \setminus K) \cap (K \cup \{a\}) = \{a, p\}\) must be connected, which is impossible. Hence if we consider the component \( M \) of \( y \) in \( X \setminus f^{-1}(C_{f(x)}) \) then \( f(M) \subseteq K \) which is also impossible because \( M \cap f^{-1}(C_{f(x)}) \neq \emptyset \).

In case (2) it can be proved in the same manner as in case (1) that \((Z \setminus K) \cap K \) is a singleton and that \( Z \setminus K \) is closed. We set \((Z \setminus K) \cap K = \{q\}\). Since \( K = K \cup \{q\}\) it follows that \((Z \setminus K) \setminus \{q\}\) is open which implies that \( q \) is a cut point of the space \( Z \). Since \( q \in Z \setminus K \) it follows that either
If $q = f(x)$ we consider again the component $M$ of $y$ in $X \setminus f^{-1}(C_{f(x)})$, and let $a \in M \cap f^{-1}(C_{f(x)})$. Then $f(M) \subseteq K$, the point $f(a)$ is a limit point of $K$ and $f(a) \in C_{f(x)}$. That is $f(a) = q$. But then there exists an open connected neighbourhood $U_a$ of $a$ such that $f(U_a) \subseteq U(f(x))$ which implies that $f(U_a) \subseteq C_{f(x)}$. Hence $U_a \subseteq f^{-1}(C_{f(x)})$ which is impossible because $U_a \cap M \neq \emptyset$. If $q \neq f(x)$ then obviously $q \in E(f(x), f(y))$.

Finally, observing that case (3) is reduced to case (1) or (2) we conclude that $E(f(x), f(y)) \neq \emptyset$, which is impossible by Proposition 1.

REFERENCES

3. Gruenhage, G. Spaces in which the non-degenerate connected subsets are the cofinite sets (to appear).