NECESSARY AND SUFFICIENT CONDITIONS FOR THE OSCILLATION OF DELAY DIFFERENTIAL EQUATION WITH A PIECEWISE CONSTANT ARGUMENT

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ABSTRACT: The characteristic equation for an equation with continuous and piecewise constant argument in the form
\[ \dot{x}(t) + p x(t - \tau) + q x([t - k]) = 0 \]
where \( p, q \in \mathbb{R} \), \( \tau \in \mathbb{R}^+ \) and \( k \in \mathbb{N} \).

is presented, which when \( q = 0 \) reduces to
\[ f(\lambda) = \lambda + e^{-\lambda \tau} = 0 \]
and when \( p = 0 \) reduces to
\[ \lambda - 1 + q \lambda^{-k} = 0. \]

Also, the necessary and sufficient conditions for oscillation are obtained.

KEY WORDS: Oscillations, Delay differential equations.


1. INTRODUCTION

The study of equations with piecewise constant argument was originated by the work of Wiener and his collaborators. See [1,2,3,4,5 and 6] and the references cited therein. In addition to its own interest this area has stimulated much activity in the study of delay difference equations.

As usual, a solution \( x(t) \) is called oscillatory if it has arbitrarily large zeros. Otherwise, the solution is called nonoscillatory. An equation is called oscillatory if all its solutions are oscillatory.

Let \([\cdot]\) denote the greatest-integer function, \( \mathbb{N} \) the set of non-negative integers and \( \mathbb{R} \) the set of real numbers.

Consider
\[ \dot{x}(t) + p x(t - \tau) + q x([t - k]) = 0 \]
where \( p, q \in \mathbb{R} \), \( \tau \in \mathbb{R}^+ \) and \( k \in \mathbb{N} \).
By a solution of Eqn.(1.1), we mean a function \( x \) which is defined on the set \([-r, \ldots, -1, 0) \cup [-\infty, \infty)\) and satisfies the following properties:

(a) \( x \) is continuous on \([-\infty, \infty)\).

(b) the derivative \( \dot{x} \) exists at each point \( t \in (0, \infty) \) with the possible exception of the points \( t \in \mathbb{N} \), where one side derivatives exist.

(c) Eqn.(1.1) is satisfied on each interval \([n, n+1)\) for \( n \in \mathbb{N} \).

Let \( \phi \in C([-r, 0], \mathbb{R}) \) and \( a_1, \ldots, a_n, a_0 \) be given real numbers such that

\[
a_{-j} = \phi(-j) \quad j \leq r, \quad j=0,1,2,\ldots,k,
\]

then one can show that Eqn. (1.1) has a unique solution satisfying the initial conditions

\[
x(t) = \phi(t) \quad -r \leq t \leq 0 \quad (1.3a)
\]

\[
x(-j) = a_{-j} \quad j=0,1,\ldots,k. \quad (1.3b)
\]

When \( q = 0 \), Eqn. (1.1) reduces to

\[
\dot{u}(t) + pu(t-r) = 0 \quad (1.4)
\]

which is oscillatory if and only if its characteristic equation

\[
f(\lambda) = \lambda + e^{-\lambda r} = 0 \quad (1.5)
\]

has no real roots, or equivalently, to

\[
p \tau > \frac{1}{e}. \quad (1.6)
\]

On the other hand, when \( p=0 \), Eqn.(1.1) reduces to

\[
\dot{v}(t) + q v([t-k]) = 0 \quad (1.7)
\]

which is oscillatory if and only if the following equation

\[
\lambda - 1 + q \lambda^{-\tau} = 0 \quad (1.8)
\]

has no positive real roots, or equivalently,

\[
q > \frac{k^k}{(k+1)^{k+1}}, \quad k \geq 1 \quad (1.9a)
\]

\[
q \geq 1, \quad k = 0. \quad (1.9b)
\]

An open question arises (see [4], p. 223) for obtaining a characteristic equation for equation (1.1) which reduces to Eqn.(1.5) when \( q=0 \) and reduces to Eqn. (1.8) when \( p=0 \) and also obtaining the necessary and sufficient conditions for oscillation of all solutions of

\[
\dot{x}(t) + px(t-1) + qx([t-1]) = 0 \quad (1.10)
\]

2. THE MAIN RESULTS

In the following, a characteristic equation associated with equation (1.1) will be presented in Theorem 2.1. Also the necessary and sufficient conditions for oscillation are obtained through Theorems 2.2 and 2.3.

THEOREM 2.1. The characteristic equation associated with equation (1.1) is

\[
f(\lambda) = \lambda - 1 + \frac{p \lambda^{-\tau}}{\ln \lambda} (\lambda - 1) + q \lambda^{-\tau} = 0 \quad (2.1)
\]
which reduces to Eqn. (1.5) when $q=0$ and reduces to Eqn. (1.8) when $p=0$.

**PROOF:** Consider Eqn.(1.1) and assume that the initial conditions (1.3a) and (1.3b) are satisfied. For $t \in [n,n+1)$, we have $[t-k] = n-k$ and one can write

\[ x(t) + px(t-\tau) + qa_{n+k} = 0, \quad t \in [n,n+1) \]  
\[ x(n) = a_n, \quad n \in \mathbb{N} \]  

Integrating (2.2a) from $n$ to $t$, we get

\[ x(t) - a_n + p \int_n^t x(s-\tau) \, ds + qa_{n+k} = 0. \tag{2.3} \]

By using the continuity of $x(t)$ as $t \to n+1$, we find

\[ a_{n+1} - a_n + p \int_n^{n+1} x(s-\tau) \, ds + qa_{n+k} = 0. \tag{2.4} \]

Assume that $x(t) = e^{\lambda t}$, $t \in [n,n+1)$, then from (2.4), we get

\[ f(\lambda) = e^\lambda - 1 + \frac{pe^{\lambda r}}{\lambda} = 0. \tag{2.5} \]

Putting $e^\lambda = \gamma$ in Eqn.(2.5), then

\[ F(\gamma) = \gamma - 1 + \frac{p\gamma e^{\lambda r}}{\ln \gamma} = 0. \tag{2.6} \]

and consequently Eqn.(2.6) has no positive real roots if and only if Eqn.(2.5) has no real roots. Assume that Eqn.(2.5) has no real roots, then $\lambda \neq 0$, and consequently $\gamma \neq 1$. If $p = 0$, then Eqn. (2.6) reduces to Eqn.(1.8), also if $q=0$, then Eqn.(2.6) reduces to Eqn.(1.5).

**THEOREM 2.2.** Equation (1.1) is oscillatory if and only if its characteristic equation (2.6) has no positive real roots.

**PROOF:** Assume that the characteristic equation (2.6) has a positive real root $\gamma_0$, then $\gamma_0'$ is a solution of Eqn.(2.4) which is a nonoscillatory solution and consequently Eqn.(1.1) is not oscillatory. On the other hand, assume that $x(t) > 0 \quad \forall \quad t \in [n,n+1)$ for sufficiently large $n$ and $F(\gamma)$ has no positive real roots. As $F(\infty) = \infty$, it follows that $F(\gamma) > 0 \quad \forall \gamma \in (0,\infty)$. For seeking the contradiction, choose:

(i) $p \leq 0$ and $q \leq 0$ then $F(\gamma) < 0 \quad \forall \gamma \in (0,1)$,
(ii) $p \geq 0$, $q < 0$ with $p < |q|$ and $\tau \leq k$ then $F(\gamma) < 0 \quad \forall \gamma \in (0,1)$,
(iii) $p < 0$, $q \geq 0$ with $q < |p| (1 - 1/e)$ and $\tau = k$ then $F(1/e) < 0$,
(iv) $p \geq 0$, $q \geq 0$ with $p + q \leq 1/8e^k$ and $\tau \leq k$ then $F(1/e) < 0$,

which is a contradiction.

**THEOREM 2.3.** If $p, q \in \mathbb{R}^+$, then all solutions of equation (1.1) are oscillatory if and only if

\[ pet + q \frac{(k+1)^{k+1}}{k^{k+1}} > 1, \quad k \geq 1 \tag{2.7} \]

**PROOF:** Assume that Eqn.(1.1) has a nonoscillatory solution, then the characteristic Eqn.(2.6) has a positive real root $\gamma_0 \in (0,1)$. Otherwise $F(\gamma_0) > 0 \quad \forall \gamma_0 \in [1,\infty)$ and therefore, we have
\[ F(y_0) = y_0 - 1 + \frac{p y_0 e^{-r}}{\ln y_0} (y_0 - 1) + q y_0^{-k} = 0, \quad y_0 \in (0, 1) \]

and then

\[ 0 = (y_0 - 1) \left(1 + \frac{p y_0 e^{-r}}{\ln y_0} + q y_0^{-k} / (y_0 - 1)\right) \]

\[ 0 = 1 + \frac{p y_0 e^{-r}}{\ln y_0} + q y_0^{-k} / (y_0 - 1) \]

\[ \leq 1 - p e^r - q \frac{(k+1)^{k+1}}{k^k} \]

which is a contradiction. On the other hand, assume that

\[ p e^r + q \frac{(k+1)^{k+1}}{k^k} \leq 1, \quad k \geq 1. \]

Now, we study the following cases:

(1) \( q = 0, p > 0 \).

Since \( F(y) > 0, \forall y \in (1, \infty) \) and \( F(e^{-r}) \leq 0 \), then there exists \( y_1 \in \mathbb{R}^+ \) such that \( F(y_1) = 0 \), i.e., the characteristic equation has a positive real root and consequently equation (1.1) is not oscillatory.

(2) \( p = 0, q > 0 \).

In this case, \( F(y) > 0, \forall y \in (1, \infty) \) and \( F\left(\frac{k}{k+1}\right) \leq 0 \). Therefore, the characteristic equation has a positive real root and then equation (1.1) has a nonoscillatory solution.

(3) \( p > 0, q > 0 \).

Since \( p e^r + q \frac{(k+1)^{k+1}}{k^k} \leq 1, \)

then,

\[ q \frac{(k+1)^{k+1}}{k^k} + p \frac{(k+1)^{k+1}}{k} \leq p e^r + q \frac{(k+1)^{k+1}}{k^k} \leq 1, \quad k \geq 1. \quad (2.8) \]

It is clear that the characteristic equation has no real roots in \((1, \infty)\) and \( F(y) > 0 \), but

\[ F\left(\frac{k}{k+1}\right) = -\frac{1}{k+1} + p \frac{(k+1)^{k+1}}{k} + q \frac{(k+1)^{k+1}}{k^k} \]

\[ = -\frac{1}{k+1} + q \frac{(k+1)^{k+1}}{k^k} + p \frac{(k+1)^{k+1}}{k} \]

From (2.8), it follows that \( F\left(\frac{k}{k+1}\right) \leq 0 \) and consequently equation (1.1) has a nonoscillatory solution.

REMARC. If \( r = k = 1 \) and \( p, q \in \mathbb{R}^+ \), then \( pe + 4q > 1 \) is a necessary and sufficient condition for oscillation of

\[ \ddot{x}(t) + p x(t-1) + q x(t-1) = 0. \]
REFERENCES


