POSITIVE SOLUTIONS OF AN ASYMPTOTICALLY PLANAR SYSTEM OF ELLIPTIC BOUNDARY VALUE PROBLEMS

F.J.S.A. CORRÊA
Departamento de Matemática e Estatística
Universidade Federal da Paraíba
58109-970-Campina Grande-PB, BRASIL
e-mail julio@dme.ufpb.br

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ABSTRACT. We will prove an existence result of positive solutions for an asymptotically planar system of two elliptic equations. It will be used as main tools for a Maximum Principle and a result on Bifurcation Theory.

KEY WORDS AND PHRASES: Positive solution, asymptotically planar, maximum principle, bifurcation point.

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1. INTRODUCTION

In this paper we will prove the existence of positive solutions for the elliptic system

\[- \Delta U = A(x)U + F(x, U) \quad \text{in } \Omega, \quad U = 0 \quad \text{on } \partial \Omega\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain, \(A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}\) whose entries are continuous in \(\Omega\),

\[U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad - \Delta U = \begin{pmatrix} - \Delta & 0 \\ 0 & - \Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} - \Delta u \\ - \Delta v \end{pmatrix} \quad \text{and } F(x, U) = \begin{pmatrix} f(x, u, v) \\ g(x, u, v) \end{pmatrix}\]

with \(f, g : \Omega \times (\mathbb{R}^+)^2 \to \mathbb{R}\) locally lipschitzian continuous satisfying

\[f(x, 0, 0) > 0 \quad \text{or } g(x, 0, 0) > 0 \quad \text{for all } x \in \Omega\]

and there is a positive constant \(C\) so that

\[0 \leq f(x, u, v), g(x, u, v) \leq C \quad \text{for all } (x, u, v) \in \Omega \times (\mathbb{R}^+)^2.\]

Condition (3) says that the function

\[\tilde{F}(x, U) = \begin{pmatrix} a(x)u + b(x)v + f(x, u, v) \\ c(x)u + d(x)v + g(x, u, v) \end{pmatrix}\]

is of asymptotically planar type. Since we are concerned with the existence of positive solutions we will suppose through this work that system (1) is cooperative, i.e., \(b(x)\) and \(c(x)\) are both nonnegative for all \(x \in \Omega\). This cooperativeness is imposed in order we may use a Maximum Principle (MP in short). In particular we will deal with the one due to the author of this paper in collaboration with M A S Souto [1]. Using this (MP) and a result on Bifurcation Theory we prove the following:

THEOREM 1. If \(a(x) < \lambda_1\), \(d(x) < \lambda_1\) and if either
(i) \[ |a|_\infty \leq |d|_\infty \text{ and } 1 < \frac{2\lambda_1}{2|d|_\infty + |b|_\infty + |c|_\infty} \]

or

(ii) \[ |d|_\infty \leq |a|_\infty \text{ and } 1 < \frac{2\lambda_1}{2|a|_\infty + |b|_\infty + |c|_\infty} \]

then problem (1) possesses a positive (classical) solution.

Here \( | \cdot |_\infty \) denotes the usual sup norm, that is, \( |u|_\infty = \sup_{x \in \Omega} |u(x)| \) and \( \lambda_1 \) is the first eigenvalue of \(-\Delta\) in \( \Omega \) under Dirichlet boundary condition.

To tackle this theorem we proceed as follows: Since \( f \) and \( g \) are both defined only for \( u, v > 0 \) we ought to consider the extensions of \( f \) and \( g \), respectively

\[
f_1(x, u, v) = f(x, |u|, |v|) \quad \text{and} \quad g_1(x, u, v) = g(x, |u|, |v|)
\]

now defined for all \((x, u, v) \in \overline{\Omega} \times \mathbb{R}^2\). We now carry on by setting

\[
L = \left( \begin{array}{cc}
-\Delta - a(x) & 0 \\
0 & -\Delta - d(x)
\end{array} \right), \quad \bar{F}(x, U) = \left( \begin{array}{c}
f_1(x, u, v) \\
g_1(x, u, v)
\end{array} \right), \quad B(x) = \left( \begin{array}{cc}
f(x) & 0 \\
c(x) & 0
\end{array} \right).
\]

Fixing these notations we are going to pay attention to the following nonlinear eigenvalue problem

\[
LU = \lambda [B(x)U + \bar{F}(x, U)] \text{ in } \Omega, \quad U = 0 \text{ on } \partial \Omega
\]

where \( \lambda > 0 \) is a real parameter and it will be proved the existence of a continuum \( \Sigma \subset \mathbb{R}^+ \times \left[ C(\Omega) \right]^2 \) of solutions \((\lambda, U)\) of (4) that begins at \((0, 0)\) and extends beyond the line \( \{1\} \times \left[ C(\Omega) \right]^2 \) arising a solution of (1) which in view of the \((MP)\), should be positive.

As we will show after proving Theorem 1 the motivation in studying problem (1) came of the scalar one

\[
-\Delta u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]

where \( f \) has a sublinear behavior.

2. PRELIMINARY RESULTS

In order to establish the \((MP)\) we begin by fixing some notations. Let \( X \) be a Banach space ordered by the positive cone \( K \subset X \) and \( \bar{L} : X \to X \) a linear operator. By a \((MP)\) to problem

\[
U = \bar{L}U + F, \quad U \in X,
\]

we mean the statement \( F \geq 0 \) (i.e. \( F \in K \)) imply \( U \geq 0 \) whenever that \( U \) is a solution of (6).

**Proposition 2 (Maximum Principle).** Let \( \bar{L} : X \to X \) be a positive linear compact operator (positive means \( \bar{L}(K) \subset K \)). Then (6) satisfies the \((MP)\) if the condition below holds true

\[
\{U \in X, t \in [0, 1], U = t\bar{L}U \} \Rightarrow U = 0.
\]

Now we shall focus our attention on the problem

\[
LU = B(x)U + \bar{F}(x, U) \text{ in } \Omega, \quad U = 0 \text{ on } \partial \Omega,
\]

to prove the following:

**Theorem 3.** If \( a(x) < \lambda_1, d(x) < \lambda_1 \) and if either

(i) \[ |a|_\infty \leq |d|_\infty \text{ and } 1 < \frac{2\lambda_1}{2|d|_\infty + |b|_\infty + |c|_\infty} \]

or
(ii) \[ |d|_{\infty} \leq |a|_{\infty} \text{ and } 1 < \frac{2\lambda_1}{2|a|_{\infty} + |b|_{\infty} + |c|_{\infty}} \]

then every solution of (8) is positive and so is a solution of (1).

**PROOF.** We first observe that the extension \( \overline{F}(x, U) \) is also nonnegative. Second we notice that the operator \( L = \begin{pmatrix} -\Delta - a(x) & 0 \\ 0 & -\Delta - d(x) \end{pmatrix} \) has an inverse

\[
L = \begin{pmatrix} (-\Delta - a(x))^{-1} & 0 \\ 0 & (-\Delta - d(x))^{-1} \end{pmatrix} : [C(\overline{\Omega})]^2 \to [C(\overline{\Omega})]^2
\]

which is compact and positive in view of \( a(x), d(x) < \lambda_1 \) in \( \overline{\Omega} \). So we will analyze uniqueness for the problem

\[ U = tL^{-1}B(x)U, \quad U \in [C(\overline{\Omega})]^2, \quad t \in [0, 1] \]

that is equivalent to

\[
\begin{cases}
-\Delta u - a(x)u = tb(x)v & \text{in } \Omega \\
-\Delta v - d(x)v = tc(x)u & \text{in } \Omega \\
u = v = 0 & \text{on } \partial\Omega
\end{cases}
\]

By multiplying both sides of the first equation in (9) by \( u \) and both sides of the second one by \( v \) and integrating by parts we obtain

\[
\int |\nabla u|^2 = \int a(x)u^2 + t \int b(x)uv
\]

and

\[
\int |\nabla v|^2 = \int c(x)uv + \int d(x)v^2
\]

Since \( a, b, c \) and \( d \) belong to \( C(\overline{\Omega}) \) one gets, thanks to both Holder's and Poincare's inequalities,

\[
\int |\nabla u|^2 \leq \frac{|a|_{\infty}}{\lambda_1} \int |\nabla u|^2 + \frac{|b|_{\infty}}{2\lambda_1} \left( \int |\nabla u|^2 + |\nabla u|^2 \right)
\]

and

\[
\int |\nabla v|^2 \leq \frac{|c|_{\infty}}{2\lambda_1} \left( \int |\nabla u|^2 + |\nabla u|^2 \right) + \frac{|d|_{\infty}}{\lambda_1} \int |\nabla v|^2
\]

Summing up these two inequalities and assuming that \( |a|_{\infty} \leq |d|_{\infty} \) one has

\[
\int |\nabla u|^2 + \int |\nabla v|^2 \leq \frac{1}{\lambda_1} \left[ \frac{2|d|_{\infty} + |b|_{\infty} + |c|_{\infty}}{2} \right] \left( \int (|\nabla u|^2 + |\nabla v|^2) \right)
\]

Since \( 1 < \frac{2\lambda_1}{2|a|_{\infty} + |b|_{\infty} + |c|_{\infty}} \) we conclude that \( U = 0 \). We arrive at the same conclusion by assuming assumption (ii). Thus system (8) enjoys the (MP) and in view of \( F(x, U) > 0 \) we have \( U \geq 0 \) and so it is a solution of (1).

We now enunciate a proposition, due to Rabinowitz [5], which is another tool in proving Theorem 1.

**PROPOSITION 4.** Let \( X \) be a Banach space and suppose that \( T : \mathbb{R}^+ \times X \to X \) is a continuous map. Then the nonlinear eigenvalue problem \( u = T(\lambda, u) \) possesses an unbounded continuum of solutions meeting \( (0, 0) \in \mathbb{R} \times X \), if in addition, we suppose \( T(0, u) = 0 \) for all \( u \in X \).

3. MAIN RESULTS AND REMARKS

We start this section proving Theorem 1.

**PROOF OF THEOREM 1.** Set \( X = [C(\overline{\Omega})]^2 \) endowed with the usual norm \( |U|_{\infty} = |u|_{\infty} + |v|_{\infty} \).

Hence \( X \) is a Banach space and, as we said before, \( L^{-1} : X \to X \) is linear, compact and positive. So problem (4) is equivalent to the following functional equation in \( \mathbb{R}^+ \times X \):
$U = \lambda [L^{-1} A(x)U + L^{-1} \bar{F}(x, U)], \, \lambda \geq 0, \, U \in X,$

(10)

where $\bar{F}(x, U)$ is the Nemytskii operator associated with the function $\bar{F}$, i.e., for each $U \in X$ one has

$\bar{F}(\cdot, U(\cdot))(x) = \bar{F}(x, U(x))$.

Since $L^{-1} A$ and $L^{-1} \bar{F}$ are compact operators we are able to conclude the existence of an unbounded continuum $\Sigma$ of solutions of (10) beginning at $(0, 0) \in \mathbb{R}^+ \times X$. If $(\lambda, 0) \in \Sigma$ then $\lambda = 0$ because $f(x, 0, 0) > 0$ or $g(x, 0, 0) > 0$. Plainly $(0, U) \in \Sigma$ implies $U = 0$. Thus $\Sigma$ meets $\{0\} \times [C(\Omega)]^2$ and $\mathbb{R}^+ \times \{0\}$ only at $(0, 0)$. Note that bootstrapping these solutions, that at first sight belong only to $[C(\Omega)]^2$, we obtain classical solutions.

It is worthy to say that hitherto we cannot affirm that $\Sigma$ contains positive solutions. In spite of this we can say that a piece (or perhaps pieces) of $\Sigma$ contains only positive solutions. Indeed, if $\lambda \leq 1$ we may prove, reasoning as in the proof of Theorem 3, that every solution $U$ of problem (10) is positive. It rests to show that in fact $\Sigma$ reaches $\lambda = 1$.

Since $\Sigma$ is unbounded it may be unbounded with respect to $\lambda$, or with respect to $U$ or with respect to both $\lambda$ and $U$. If $\Sigma$ is unbounded in $\lambda$ then it crosses the line $\{1\} \times X$ and so we find a solution $U$ of the problem (4) and in view of assumptions (i) and (ii) of Theorem 3 is positive and so is a solution of (1). We now suppose that if $(\lambda, U) \in \Sigma$ then $\lambda \leq 1$. Hence there is a sequence $(\lambda_n, U_n) \in \Sigma$ with $\lambda_n \leq 1$ and $|U_n|_\infty \to \infty$. Thus

$L U_n = \lambda [B(x)U_n + \bar{F}(x, U_n)]$ in $\Omega, \, U_n = 0$ on $\partial \Omega$.

Setting $W_n = \frac{U_n}{|U_n|_\infty}$ we obtain

$L W_n = \lambda_n [B(x)W_n + \frac{\bar{F}(x, U_n)}{|U_n|_\infty}]$ in $\Omega, \, W_n = 0$ on $\partial \Omega$.

Passing to a subsequence if necessary we obtain $\lambda_n \to \lambda_0 \in [0, 1], \, W_n \to W$ in $[C(\Omega)]^2$ and $L W_n \to \lambda_0 B(x)W$. As $L : D(L) \to [C(\Omega)]^2$, where $D(L) = \{U \in [C(\Omega)]^2; \, \text{L} \in [C(\Omega)]^2$ and $U = 0$ on $\partial \Omega\}$, is closed one has that $W \in D(L)$ and

$L W = \lambda_0 A(x)W$ in $\Omega, \, W = 0$ on $\partial \Omega$.

Because $W_n \to W$ in $[C(\Omega)]^2$ and $|W_n|_\infty = 1$ then $|W|_\infty = 1$, i.e., $W$ is a nontrivial solution of the above problem. But, in view of $(MP)$ and $\lambda_0 \leq 1, \, W \equiv 0$ which is absurd. Thus $\Sigma$ crosses $\{1\} \times X$ and, by Theorem 3, such solution is positive and the proof of Theorem 1 is over.

**Remark 1.** The proof of Theorem 1 rests heavily on the existence of a $(MP)$ like the one contained in [1]. We must observe that this $(MP)$ is valid for a more general elliptic operator. Indeed, if we consider uniformly elliptic operators in the divergence form

$L_k = -D_j(a^k_j D_i u) + D_j(a^k_j u), \, k = 1, 2$

(the symbols of summation are implicit in the expressions) where coefficients are regular enough, $a^k_j = a^k_j$ and setting $\lambda_1(L_k)$ as being the first eigenvalue of $(L_k, H_0^1(\Omega))$, the system

$L U = A(x)U + F(x)$ in $\Omega, \, U = 0$ on $\partial \Omega$

where

$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$, enjoys the $(MP)$ if

$\langle B(x)\xi, \xi \rangle < \lambda_1(L_k)(\xi_1^2 + \xi_2^2)$
for all $\xi = (\xi_1, \xi_2)$ and $x \in \bar{\Omega}$. Here $(\cdot, \cdot)$ is the usual inner product in $\mathbb{R}^2$ and $B(x) = \begin{pmatrix} 0 & b(x) \\ c(x) & 0 \end{pmatrix}$.

Note that the above condition provides the uniqueness required by the (MP) in \cite{1}. So Theorems 1 and 3 remain valid, with slight modifications, if system (1) is considered with $-\Delta$ substituted by the nonselfadjoint operators $L_1$ and $L_2$.

**REMARK 2.** At the outset of our motivations in studying problem (1) we had considered the following

$$
Lu := -\sum_{i,j=1}^N a_{ij}(x)D_{ij}u + \sum_{i=1}^N a_i(x)D_iu = f(x, u) \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega \tag{11}
$$

where $L$ is a second order uniformly elliptic operator in $\Omega$ with real smooth coefficients satisfying $a_{ij} = a_{ji}$ in $\bar{\Omega}$, for all $1 \leq i, j \leq N$, and $f: \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a sublinear nonlinearity. It is to say, setting

$$
a_0(x) = \lim_{t \to 0^+} \frac{f(x, t)}{t}, \ a_\infty(x) = \limsup_{t \to \infty} \frac{f(x, t)}{t} \tag{12}
$$

one must have

$$
\lambda_1(a_0) < 1 < \lambda_1(a_\infty) \tag{13}
$$

where $\lambda_1(a_i)$, $i = 0, \infty$, is the first eigenvalue of the linear eigenvalue problem

$$
Lu = \lambda a_i(x)u \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega.
$$

Condition (13) says that we are working with a sublinear problem, i.e., in case, for instance, $a_0$ and $a_\infty$ are constants the nonlinearity $f$ begins, above the straight line $\lambda_1 t$ and at the end it remains below the same line.

In Brezis-Oswald \cite{2} the authors consider $L = -\Delta$ and use Variational Methods by exploring the selfadjointness of $-\Delta$ and $f$ is not necessarily a smooth function. In fact $a_0(x)$ and $a_\infty(x)$ may take values $+\infty$ and $-\infty$, respectively, so we address the reader to Section 3 of \cite{2} for the precise meaning of (13).

In de Figueiredo \cite{3} problem (11) is studied under condition (13) where $L$ is a selfadjoint operator more general than $-\Delta$ but $f$ is a $C^\alpha$-function, $0 < \alpha < 1$, and $f(x, t) + Kt$ is nondecreasing in $t$ for some $K \geq 0$. In this case the sub and supersolution method is used.

If $L$ is not necessarily selfadjoint problem (11) was studied by Costa-Gonçalves \cite{4} under condition (13), still using the sub and supersolution technique. In the works quoted above the authors always show existence of a positive solution as well as give sufficient condition for uniqueness.

This scalar problem arises a very natural question: How can we formulate a sublinear problem like before when we take a system into account?

We think that the best motivation towards a more general situation is to consider the biharmonic problem because it brings up for attention a very simple system and from it we would deal a more general sublinear problem. More precisely we first analyze the simplest biharmonic problem

$$
\Delta^2 u = mu + g(u) \text{ in } \Omega, \ u = \Delta u = 0 \text{ on } \partial \Omega, \tag{14}
$$

that is, the biharmonic equation under the so called Navier boundary conditions. Here $m$ is a positive constant. Setting $v = -\Delta u$, $A = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$, $G(U) = \begin{pmatrix} 0 \\ g(u) \end{pmatrix}$ we get the system

$$
-\Delta U = AU + G(U) \text{ in } \Omega, \ U = 0 \text{ on } \partial \Omega. \tag{15}
$$
Taking $g$ a bounded function then $f(u) = mu + g(u)$ would be sublinear if it begins at $t = 0$ zero above $\lambda_1^2 t$ and remains below $\lambda_1^2 t$ for $t$ large enough. Note that this is the counterpart of condition (13) when we are dealing with $\Delta^2$. Observe that $\lambda_1^2$ is the first eigenvalue of $\Delta^2$ in $\Omega$ under Navier boundary conditions and the situation described above occurs, for instance, if $g(0) > 0$ and $m < \lambda_1^2$.

**REMARK 3.** Now we are going to analyze the condition given in Theorem 1 for the system (15). In this case one has that $\frac{1+m}{2} < \lambda_1$ is a sufficient condition in order system (15) enjoys the (MP).

Next we will show that this condition leads to a sublinear problem related to (14). Let us suppose that $\frac{1+m}{2} < \lambda_1$

a) If $m \neq 1$ one has $(m - 1)^2 > 0$ which implies $\frac{(1+m)^2}{4} > m$ and since $\lambda_1 > \frac{1+m}{2}$ we get $\lambda_1^2 > \frac{(1+m)^2}{4} > m$ and so we have a sublinear problem.

b) If $m = 1$ then $\lambda_1 > \frac{1+m}{2} = 1$ and hence $\lambda_1^2 > \lambda_1 > 1 = m$. In this case we still have a sublinear problem.

Therefore we believe that conditions (i) and (ii) are two kinds of sublinearity conditions when we deal with a system of two equations.

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