SOME RESULTS ON DOMINANT OPERATORS

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ABSTRACT. We show that the Weyl spectrum of a dominant operator satisfies the spectral
mapping theorem for analytic functions and then answer a question of Oberai.

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1. INTRODUCTION

Throughout this paper $H$ will denote an infinite dimensional Hilbert space and $B(H)$ the
space of all bounded linear operators on $H$. If $T \in B(H)$, we write $\sigma(T)$ for the spectrum of
$T$, $\pi_0(T)$ for the set of eigenvalues of $T$, and $\pi_{\infty}(T)$ for the isolated points of $\sigma(T)$ that are
eigenvalues of finite multiplicity. If $K$ is a subset of $\mathbb{C}$, we write $\text{iso } K$ for the set of isolated
points of $K$. An operator $T \in B(H)$ is said to be Fredholm if its range $\text{ran } T$ is closed and both
the null space $\ker T$ and $\ker T^*$ are finite dimensional. The index of a Fredholm operator $T$,
denoted by $i(T)$, is defined by

$$i(T) = \dim \ker T - \dim \ker T^*.$$  

The essential spectrum of $T$, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm} \}.$$  

A Fredholm operator of index zero is called a Weyl operator. The Weyl spectrum of $T$, denoted
by $\omega(T)$, is defined by

$$\omega(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl} \}.$$  

It was shown by Berberian [2] that $\omega(T)$ is a nonempty compact subset of $\sigma(T)$.

An operator $T \in B(H)$ is said to be dominant if for every $z \in \mathbb{C}$ there exists a real number
$M_z > 0$ such that

$$(T - z)(T - z)^* \leq M_z(T - z)^*(T - z) \quad (1.1)$$  

In this case, if $\sup_{z \in \mathbb{C}} M_z = M < \infty$, $T$ is said to be $M$-hyponormal, and if $M = 1$, $T$ is
hyponormal. Evidently,

$$T \text{ is hyponormal } \implies T \text{ is } M\text{-hyponormal } \implies T \text{ is dominant}$$

We also note that an operator $T$ need not be hyponormal even though $T$ and $T^*$ are both
$M$-hyponormal. To see this, consider the operator

$$T = \begin{bmatrix} U & K \\ 0 & U^* \end{bmatrix} : l_2 \oplus l_2 \to l_2 \oplus l_2,$$
where $U$ is the unilateral shift on $l_2$ and $K : l_2 \to l_2$ is given by

$$K(x_1, x_2, x_3, \cdots) = (2x_1, 0, 0, \cdots).$$

Then a direct calculation shows that

$$\frac{1}{2} \|(T - z)x\| \leq \|(T - z)^*x\| \leq 2\|(T - z)x\|$$

for all $z \in \mathbb{C}$ and for all $x \in l_2 \oplus l_2$, which says that $T$ and $T^*$ are both dominant (even $M$-hyponormal). But since

$$[I \quad 0] + [0 \quad \frac{3}{2}K] = T^*T \neq TT^* = [I \quad \frac{3}{2}K \quad 0]$$

$T$ is not normal (even hyponormal).

If $T$ is Fredholm then by (1.1)

$$T \text{ dominant } \implies i(T) \leq 0. \quad (1.2)$$

It was known by Oberai [8] that the mapping $T \to \omega(T)$ is upper semi-continuous, but not continuous at $T$. However if $T_n \to T$ with $T_n T = T T_n$ for all $n \in \mathbb{N}$ then

$$\lim \omega(T_n) = \omega(T). \quad (1.3)$$

It was known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic functions: if $f$ is analytic on a neighborhood of $\sigma(T)$ then

$$\omega(f(T)) \subset f(\omega(T)). \quad (1.4)$$

The inclusion (1.4) may be proper (see Berberian [2, Example 3.3]). If $T$ is normal then $\sigma_s(T)$ and $\omega(T)$ coincide. Thus if $T$ is normal since $f(T)$ is also normal, it follows that $\omega(T)$ satisfies the spectral mapping theorem for analytic functions. We say that Weyl's theorem holds for $T$ if

$$\omega(T) = \sigma(T) - \pi_{00}(T).$$

It was known (Berberian [1]) that Weyl's theorem holds for any hyponormal operator — indeed, for any seminormal operator and for any Toeplitz operator. Oberai [9] has raised the following question: Does there exist a hyponormal operator $T$ such that Weyl's theorem does not hold for $T^2$? Note that $T^2$ may not be hyponormal even if $T$ is hyponormal (Halmos [5, Problem 209]).

In this paper we show that the Weyl spectrum of a dominant operator satisfies the spectral mapping theorem for analytic functions, and that Weyl's theorem holds for $p(T)$ when $T$ is hyponormal and $p$ is any polynomial. The latter result answers the question of Oberai.

2. SPECTRAL MAPPING THEOREM

THEOREM 2.1. If $S$ and $T$ are dominant operators, then

$$S, T \text{ Weyl } \iff ST \text{ Weyl.} \quad (2.1)$$

PROOF. If $S, T$ are Weyl, then $S, T$ are Fredholm and $i(S) = i(T) = 0$. By Conway [3], $ST$ is Fredholm and by the index product theorem, $i(ST) = i(S) + i(T) = 0$. Hence $ST$ is Weyl.

Conversely if $ST$ is Weyl, then $ST$ is Fredholm and $i(ST) = 0$. Since $S$ and $T$ are dominant, $\ker S \subset \ker S^*$ and $\ker T \subset \ker T^*$. Since $\ker S^* \subset \ker(ST)^*$, $\dim \ker S \leq \dim \ker S^* \leq \dim \ker T$. Therefore $ST$ is Weyl.
dim \text{ker}(ST)^* < \infty. \text{ Thus ker} S \text{ and ker} S^* \text{ are finite dimensional. By Schechter [10, Chap. 5 Theorem 3.5], S and T are Fredholm. Since } 0 = i(ST) = i(S) + i(T) \text{ by the index product theorem, by (1.2) } i(S) = i(T) = 0. \text{ Hence S and T are Weyl.}

If the "dominant" condition is dropped in the above theorem, then the backward implication may fail even though \( T_1 \) and \( T_2 \) commute: For example, if \( U \) is the unilateral shift on \( l_2 \), consider the following operators on \( l_2 \oplus l_2 : T_1 = U \oplus I \) and \( T_2 = I \oplus U^* \).

**THEOREM 2.2.** If \( T \) is dominant and \( f \) is analytic on a neighborhood of \( \sigma(T) \), then \( \omega(f(T)) = f(\omega(T)) \).

**PROOF.** Suppose that \( p \) is any polynomial. Let

\[
P(T) - \lambda I = a_0(T - \mu_1I) \cdots (T - \mu_nI).
\]

Since \( T \) is dominant, \( T - \mu_iI \) are dominant operators for each \( i = 1, 2, \ldots, n \). It thus follows from Theorem 2.1 that

\[
\lambda \notin \omega(p(T)) \iff p(T) - \lambda I = \text{Weyl}
\]

\[
\iff a_0(T - \mu_1I) \cdots (T - \mu_nI) = \text{Weyl}
\]

\[
\iff T - \mu_iI = \text{Weyl for each } i = 1, 2, \ldots, n
\]

\[
\iff \mu_i \notin \omega(T) \text{ for each } i = 1, 2, \ldots, n
\]

\[
\iff \lambda \notin p(\omega(T))
\]

which says that \( \omega(p(T)) = p(\omega(T)) \). If \( f \) is analytic on a neighborhood of \( \sigma(T) \), then there is a sequence \( (p_n) \) of polynomials such that \( f_n \to f \) uniformly on \( \sigma(T) \). Since \( p_n(T) \) commutes with \( f(T) \), by Oberai [8]

\[
f(\omega(T)) = \lim p_n(\omega(T)) = \lim \omega(p_n(T)) = \omega(f(T)).
\]

Recall that \( T \in B(H) \) is said to be isoloid if \( \text{iso} \sigma(T) \subset \pi_0(T) \) (Oberai [9]).

**LEMMA 2.3.** (Oberai [9]) Let \( T \in B(H) \) be isoloid. Then for any polynomial \( p(t) \),

\[
p(\sigma(T) - \pi_0(T)) = \sigma(p(T)) - \pi_0(p(T)).
\]

Let \( T \) be an \( M \)-hyponormal operator which satisfies the additional property that for all \( z \) in the complex plane, all integers \( n \) and all \( x \) in \( H \),

\[
\|(T - z)^nx\|^2 < M\|(T - z)^2n\|^2 \cdot \|x\|^2.
\]

\( T \) is said to be an operator of \( M \)-power class (N) (Istrătescu [7]). The following \( M \)-hyponormal operator \( T \) which is not hyponormal is of \( M \)-power class (N) (Istrătescu [7]): Let \( \{e_i\} \) be an orthonormal basis for \( H \), and define

\[
Te_i = \begin{cases} 
e_2, & \text{if } i = 1 \\ 2e_3, & \text{if } i = 2 \\ e_{i+1}, & \text{if } i \geq 3 \end{cases}
\]

i.e., \( T \) is a weighted shift. From the definition of \( T \) we see that \( T \) is similar to the unilateral shift \( U \) (Halmos [5], Problem 90). Thus there exists an \( S \) such that \( T = SUS^{-1} \). In our case \( \|S\| = 2 \), \( \|S^{-1}\| = 1 \). Since \( U \) is the unilateral shift, \( U \) is a hyponormal operator, and thus for every \( n \) and \( z \in \mathbb{C} \) the operator \( (U - z)^n \) is of class \( (N) \). It follows that

\[
\|(U - z)^nx\|^2 \leq \|(U - z)^{2n}\|^2 \cdot \|x\|^2
\]
for all $x \in H$ with $\|x\| = 1$, and hence $T$ is of $M$–power class with $M = 4$. Thus our class is strictly larger than the class of hyponormal operators. Since $w(T) = w(U) = D$ (the closed unit disc) and $\pi_0(T) = \emptyset$, $\sigma(T) = w(T)$ and so Weyl’s theorem holds for $T$.

**THEOREM 2.4.** If $T \in B(H)$ is an operator of $M$–power class $(N)$, then for any polynomial $p$ on a neighborhood of $\sigma(T)$ Weyl’s theorem holds for $p(T)$.

**PROOF.** By Istratescu [7], $T$ is isoloid and Weyl’s theorem holds for any operator of $M$–power class $(N)$. Hence by Theorem 2.2 and Lemma 2.3,

$$w(p(T)) = p(w(T)) = p(\sigma(T) - \pi_0(T)) = \sigma(p(T)) - \pi_0(p(T))$$

Therefore Weyl’s theorem holds for $p(T)$.

Since every hyponormal operator is of 1-power class $(N)$, we obtain the following result which answers the question of Oberai.

**COROLLARY 2.5.** If $T \in B(H)$ is hyponormal, then for any polynomial $p$ on a neighborhood of $\sigma(T)$ Weyl’s theorem holds for $p(T)$.

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**REFERENCES**


