ON NEW STRENGTHENED HARDY-HILBERT'S INEQUALITY

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ABSTRACT. In this paper, a new inequality for the weight coefficient \( \omega(q, n) \) in the form

\[
\omega(q, n) := \sum_{m=1}^{\infty} \frac{1}{m + n} \left( \frac{n}{m} \right)^{1/q} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}} \left( q > 1, \frac{1}{p} + \frac{1}{q} = 1, n \in N \right)
\]

is proved. This is followed by a strengthened version of the Hardy-Hilbert inequality.

KEY WORDS AND PHRASES: Hardy-Hilbert's inequality, weight coefficient, Holder's inequality.

1. INTRODUCTION

If \( a_n \geq 0, 0 < \sum_{n=1}^{\infty} n^2 a_n^2 < \infty \), then the Karlson's inequality is

\[
\left( \sum_{n=1}^{\infty} a_n \right)^4 < \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} n^2 a_n^2,
\]

where the constant \( \pi^2 \) cannot be made smaller. However, it can be strengthened (see Miklin [1], p. 7) as

\[
\left( \sum_{n=1}^{\infty} a_n \right)^4 < \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} \left( n - \frac{1}{2} \right)^2 a_n^2.
\]

In recent years, considerable attention has been given to develop some types of strengthened inequality (see [2]-[10]) by estimating the weight coefficient \( \omega(q, n) \) as

\[
\omega(q, n) = \sum_{m=1}^{\infty} \frac{1}{(m + n)} \left( \frac{n}{m} \right)^{1/q} \left( q > 1, p^{-1} + q^{-1} = 1, n \in N \right).
\]

Some improvement of Hardy-Hilbert's inequality (see Hardy et al. [11]) has been made in the form

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m + n} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}.
\]

In their recent work, Xu and Gau [2] considered the following weight coefficient (1.3) and proved the following inequality
$$\omega(q, n) < \frac{\pi}{\sin(\pi/p)} - \frac{\eta_p}{n^{1/p} + n^{-1/q}}, \quad \eta_p = p - 1. \quad (1.5)$$

Then a strengthened Hardy-Hilbert's inequality

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left\{ \sum_{n=1}^{\infty} \left[ \frac{\pi}{\sin(\pi/p)} - \frac{p-1}{n^{1/p} + n^{-1/q}} \right] a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \left[ \frac{\pi}{\sin(\pi/p)} - \frac{q-1}{n^{1/q} + n^{-1/p}} \right] b_n^q \right\}^{1/q} \quad (1.6)$$

was proved. The key is to estimate the corresponding weight coefficient effectively. Hsu and Wang [3] proved the following inequality

$$\omega(2, n) < \pi - \frac{\theta}{\sqrt{n}}, \quad \theta = \frac{3}{\sqrt{2}} - 1 = 1.12132^+ \quad (n \in N). \quad (1.7)$$

Then they gave a new strengthened Hilbert's inequality which is the same as (1.6) with $p = 2$. Since $\theta$ in (1.7) is not the best possible, Gau [5] obtained the best possible value of

$$\theta = \pi - \sum_{k=1}^{\infty} \left( \frac{1}{k^2} \right) = 1.2811^+. \quad (1.7)$$

Subsequently, Gau [6] considered the general case and proved a new inequality for the weight coefficient $\omega(q, n)$ as

$$\omega(q, n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{\sqrt{n}}, \quad (q > 1, \frac{1}{p} + \frac{1}{q} = 1, n \in N) \quad (1.8)$$

where $\theta_p = (p - 1)$. Recently, Gau [7] replaced $(p - 1)$ by $\theta_p = \theta_p(n) > 0$ in (1.8). But the problem is that $\theta_p(n)$ depends on both $p$ and $q$. Simultaneously, Yang [8] found that $\theta_p = \theta = 0.341295^+$, but the constant $\theta_p = \theta$ is not the best possible value. Finally, Yang and Gau [9] found the best possible value for $\theta_p = \theta = 1 - C = 0.4227843^+$, where $C$ is an Euler constant. They also proved the following new Hardy-Hilbert's inequality

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left\{ \sum_{n=1}^{\infty} \left[ \frac{\pi}{\sin(\pi/p)} - \frac{1-C}{n^{1/p} + n^{-1/q}} \right] a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \left[ \frac{\pi}{\sin(\pi/p)} - \frac{1-C}{n^{1/q} + n^{-1/p}} \right] b_n^q \right\}^{1/q} \quad (1.9)$$

It is important to point out that (1.5) and (1.8) are different, and the constant $\eta_p$ in (1.5) depends on $p$.

The main objective of this paper is to prove an improved version of (1.5) as

$$\omega(q, n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}}, \quad (q > 1, \frac{1}{p} + \frac{1}{q} = 1, n \in N) \quad (1.10)$$

and then prove a strengthened version of Hardy-Hilbert's inequality as follows:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left\{ \sum_{n=1}^{\infty} \left[ \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}} \right] a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \left[ \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q} + n^{-1/p}} \right] b_n^q \right\}^{1/q} \quad (1.11)$$

For this, we need the following inequality (see Yang [8] Lemma 1): If

$$f(x) > 0, f^{(2r-1)}(x) < 0, f^{(2r)}(x) \geq 0, x \in [1, \infty)(r = 1, 2), f^{(r)}(\infty) = 0(r = 0, 1, 2, 3, 4),$$

and $\int_1^{\infty} f(x)dx < \infty$, then

$$\sum_{m=1}^{\infty} f(m) \leq \int_1^{\infty} f(x)dx + \frac{1}{2} f(1) - \frac{1}{12} f'(1). \quad (1.12)$$
2. SOME LEMMAS

**Lemma 2.1.** If \( q > 1, p^{-1} + q^{-1} = 1, n \in N, \) then
\[
\omega(q, n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{n^{1/p}} [f_n(p) + g_n(p)],
\]
(2.1)

where \( \omega(q, n) \) is defined by (1.5), and
\[
f_n(p) := p + \frac{1}{12p} + \frac{1}{(1 + p)n} + \frac{1}{12p^2} + \frac{1}{12p^2(1 + 3p)n^3},
\]
\[
g_n(p) := -\frac{1}{12p^2} - \frac{1}{2(1 + 2p)n^2} - \frac{7}{12} - \frac{1}{2n} + \frac{1}{12n^2} - \frac{7}{12n^3}.
\]

**Proof.** Let
\[
f(x) = \frac{1}{(x + n)x^{1/q}}, \quad x \in [1, \infty) (q > 1, n \in N).
\]
By (1.12), we obtain that
\[
\sum_{m=1}^{\infty} \frac{1}{(m + n)m^{1/q}} \leq \int_{1}^{\infty} \frac{1}{(x + n)x^{1/q}} \, dx + \left(\frac{7}{12} - \frac{1}{12p}\right) \frac{1}{1 + n} + \frac{1}{12(1 + n)^2}.
\]
(2.2)

Since
\[
\int_{0}^{1/n} \frac{1}{(1 + y)y^{1/q}} \, dy = \int_{0}^{1/n} \sum_{\nu=0}^{\infty} (-1)^\nu y^{\nu-1/q} \, dy
\]
\[
= \sum_{\nu=0}^{\infty} (-1)^\nu \int_{0}^{1/n} y^{\nu-1/q} \, dy = \frac{p}{n^{1/p}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(1 + \nu p)n^\nu}
\]
\[
> \frac{p}{n^{1/p}} \sum_{\nu=1}^{3} \frac{(-1)^\nu}{(1 + \nu p)n^\nu} = \frac{1}{n^{1/p}} \left[ p + \sum_{\nu=1}^{3} \frac{(-1)^\nu}{\nu n^\nu} - \sum_{\nu=1}^{3} \frac{(-1)^\nu}{(1 + \nu p)n^\nu} \right].
\]

Putting \( x = ny, \) we find that
\[
\int_{1}^{\infty} \frac{1}{(x + n)x^{1/q}} \, dx = \frac{1}{n^{1/q}} \int_{1/n}^{\infty} \frac{1}{(1 + y)y^{1/q}} \, dy
\]
\[
= \frac{1}{n^{1/q}} \left[ \int_{0}^{\infty} \frac{1}{(1 + y)y^{1/q}} \, dy - \int_{0}^{1/n} \frac{1}{(1 + y)y^{1/q}} \, dy \right]
\]
\[
= \frac{1}{n^{1/q}} \left[ \frac{\pi}{\sin(\pi/p)} - \frac{p}{n^{1/p}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(1 + \nu p)n^\nu} \right]
\]
\[
< \frac{1}{n^{1/q}} \frac{\pi}{\sin(\pi/p)} - \frac{1}{n} \left[ p + \sum_{\nu=1}^{3} \frac{(-1)^\nu}{\nu n^\nu} - \sum_{\nu=1}^{3} \frac{(-1)^\nu}{(1 + \nu p)n^\nu} \right],
\]
we then find that
\[
\frac{1}{1 + n} = \frac{1}{n} \left(1 + \frac{1}{n}\right)^{-1} < \frac{1}{n} \left(1 - \frac{1}{n} + \frac{1}{n^2}\right),
\]
and
\[
\frac{1}{(1 + n)^2} = \frac{1}{n^2} \left(1 + \frac{1}{n}\right)^{-2} < \frac{1}{n^2} \left(1 - \frac{2}{n} + \frac{3}{n^2}\right).
\]
Substituting the above results in (2.2), by (1.5), we have (2.1). This proves the lemma.
LEMMA 2.2. If $p > 1$, $n \in N$, then

$$f_n(p) + g_n(p) > \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}. \quad (2.3)$$

PROOF. Since

$$f_n'(p) = 1 - \frac{1 + n^2}{12n^2 p^2} - \frac{1}{(1 + p)^2 n} - \frac{1}{(1 + 3p)^2 n^3}$$

$$> 1 - \frac{1 + n^2}{12n^2} - \frac{1}{(1 + 1)^2 n} - \frac{1}{(1 + 3)^2 n^3}$$

$$= \frac{11}{12} - \frac{1}{12n^2} - \frac{1}{4n} - \frac{1}{16n^3} > 0,$$

and

$$g_n'(p) = \frac{1}{12p^2 n} + \frac{1}{(1 + 2p)^2 n^2} > 0,$$

then $f_n(p) + g_n(p)$ is strictly increasing for $p \in (1, \infty)$, and

$$f_n(p) + g_n(p) > \lim_{p \to 1} (f_n(p) + g_n(p)) = \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}.$$ 

Thus the lemma is proved.

LEMMA 2.3. If $q > 1$, $p^{-1} + q^{-1} = 1$, $n \in N$, then inequality (1.10) is valid. So is the following inequality:

$$\omega(p, n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q} + n^{-1/p}}. \quad (2.4)$$

PROOF. Since for $n \geq 3$,

$$\left(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\right)\left(1 + \frac{1}{2n}\right) = \frac{1}{2} + \frac{1}{n} \left(\frac{1}{6} - \frac{1}{24n} - \frac{1}{2n^2} - \frac{1}{4n^3}\right) > \frac{1}{2},$$

then

$$\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3} > \frac{1}{2 + n^{-1}} \quad (n \geq 3).$$

By (2.1) and (2.3), we have

$$\omega(q, n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{n^{1/p}} \left(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\right)$$

$$< \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}} \quad (n \geq 3).$$

Taking $\theta_p = 1 - C$, by (1.8) (see Yang and Gau [9]), we find that

$$\omega(q, 1) < \frac{\pi}{\sin(\pi/p)} - \frac{1 - C}{1} < \frac{\pi}{2 \times 2 + 1}. \quad (2.5)$$

Since $C < 3/5 = 0.6$, then we have

$$\frac{1}{2 \times 2^{1/p} + 2^{-1/q}} < \frac{1 - C}{2^{1/p}},$$

and

$$\omega(q, 2) < \frac{\pi}{\sin(\pi/p)} - \frac{1 - C}{2^{1/p}} < \frac{\pi}{2 \times 2^{1/p} + 2^{-1/q}}. \quad (2.6)$$

It follows that for $n = 1, 2$, (1.10) also holds. Then (1.10) is valid for any $n \in N$. Interchanging $p, q$ in (1.10), since $\frac{\pi}{\sin(\pi/p)} = \frac{\pi}{\sin(\pi/q)}$, we have (2.4). The lemma is proved.
3. MAIN RESULTS

THEOREM 3.1. If \( p > 1, p^{-1} + q^{-1} = 1, a_n \geq 0, b_n \geq 0, \) and \( 0 < \sum_{n=1}^{\infty} a_n^p < \infty, \)
\( 0 < \sum_{n=1}^{\infty} b_n^q < \infty, \) then inequality (1.11) is valid. We also have
\[
\sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{a_n}{m + n} \right)^p < \left[ \frac{\pi}{\sin(\pi/p)} \right]^{p-1} \sum_{n=1}^{\infty} \left[ \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}} \right] a_n^p. \tag{3.1}
\]

When \( p = q = 2, \) this inequality reduces to the form
\[
\sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{a_n}{m + n} \right)^2 < \pi \sum_{n=1}^{\infty} \left[ \frac{1}{2\sqrt{n + 1/n - 1}} \right] a_n^2. \tag{3.2}
\]

PROOF. By Holder's inequality, we have
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n b_n}{m + n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ \frac{1}{(m + n)^{1/p}} \left( \frac{m}{n} \right)^{1/p} a_n \right] \left[ \frac{1}{(m + n)^{1/q}} \left( \frac{n}{m} \right)^{1/q} b_n \right] \\
\leq \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m + n)^{1/p}} \left( \frac{m}{n} \right)^{1/p} a_n^p \right\}^{1/p} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m + n)^{1/q}} \left( \frac{n}{m} \right)^{1/q} b_n^q \right\}^{1/q} \\
= \left\{ \sum_{n=1}^{\infty} \omega(q, n) a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \omega(p, n) b_n^q \right\}^{1/q}.
\]
Hence, by (1.10) and (2.4), inequality (1.11) holds.

Since by (2.4), \( \omega(p, n) < \frac{\pi}{\sin(\pi/p)} \), then by Holder's inequality, we obtain
\[
\sum_{n=1}^{\infty} \frac{a_n}{m + n} = \sum_{n=1}^{\infty} \left[ \frac{1}{(m + n)^{1/p}} \left( \frac{m}{n} \right)^{1/p} a_n \right] \left[ \frac{1}{(m + n)^{1/q}} \left( \frac{n}{m} \right)^{1/q} \right] \\
\leq \left\{ \sum_{n=1}^{\infty} \frac{1}{(m + n)^{1/q}} \left( \frac{n}{m} \right)^{1/q} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \frac{1}{(m + n)^{1/p}} \left( \frac{m}{n} \right)^{1/p} \right\}^{1/q} \\
= \left\{ \sum_{n=1}^{\infty} \frac{1}{(m + n)^{1/q}} \left( \frac{n}{m} \right)^{1/q} a_n^p \right\}^{1/p} \left\{ \omega(p, n) \right\}^{1/q} \\
< \left\{ \sum_{n=1}^{\infty} \frac{1}{(m + n)^{1/q}} \left( \frac{n}{m} \right)^{1/q} a_n^p \right\}^{1/p} \left\{ \frac{\pi}{\sin(\pi/p)} \right\}^{1/q}.
\]
By (1.10), we find
\[
\sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{a_n}{m + n} \right)^p < \left[ \frac{\pi}{\sin(\pi/p)} \right]^{p/q} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m + n} \left( \frac{n}{m} \right)^{1/q} a_n^p \\
= \left[ \frac{\pi}{\sin(\pi/p)} \right]^{p-1} \sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{1}{m + n} \left( \frac{n}{m} \right)^{1/q} \right\} a_n^p \\
= \left[ \frac{\pi}{\sin(\pi/p)} \right]^{p-1} \sum_{n=1}^{\infty} \omega(q, n) a_n^p \\
< \left[ \frac{\pi}{\sin(\pi/p)} \right]^{p-1} \sum_{n=1}^{\infty} \left[ \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}} \right] a_n^p.
\]
This proves result (3.1). Thus the proof of Theorem 3.1 is complete.
4. CONCLUDING REMARKS

(a) Inequality (1.11) is a definite improvement over (1.6).

(b) Since, for \( n \geq 3 \), \( C > \left( \frac{n+1}{2n+1} \right) \), then

\[
\frac{\pi}{\sin(\pi/p)} \left( \frac{1}{2n^{1/p} + n^{-1/q}} \right) < \frac{\pi}{\sin(\pi/p)} - \frac{(1 - C)}{n^{1/p}}, \quad (n \geq 3).
\]

(4.1)

In view of (2.5), (2.6) and (3.3), it follows that (1.9) and (1.11) represent two distinct versions of strengthened inequalities. But they are not comparable.

(c) Inequality (3.1) reduces to

\[
\sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{a_n}{m+n} \right)^p < \left[ \frac{\pi}{\sin(\pi/p)} \right]^p \sum_{n=1}^{\infty} a_n^p.
\]

(4.2)

This is an equivalent form of Hardy-Hilbert's inequality (1.4) (see Hardy et al. [11], Chapter 9).

REFERENCES


